Chapter-7	Higher Order Linear Differential Equations with Constant Coefficients
Topic-7.9	

Legendre's Differential Equations

Differential Equation of the form

$(ax+b)^3\dfrac{d^3y}{dx^3}+P_1\cdot(ax+b)^2\dfrac{d^2y}{dx^2}+P_2\cdot(ax+b)\dfrac{dy}{dx}+P_3\cdot y=Q(x)$, where P_1,P_2,P_3 are constants is called Legendre's Differential Equation.

Using substitution $(ax+b)=e^z$, we can transform above differential equation into Differential Equation with constant coefficients

Put $ax+b=e^z \quad \therefore \ \log(ax+b)=z$ and $\dfrac{dz}{dx}=\dfrac{a}{ax+b}$

By chain rule, $\dfrac{dy}{dx}=\dfrac{dy}{dz}\dfrac{dz}{dx}=\dfrac{dy}{dz}\dfrac{a}{ax+b}$

$\therefore (ax+b)\dfrac{dy}{dx}=a\dfrac{dy}{dz}=aDy$, where $D=\dfrac{d}{dz}$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{a}{ax+b}\frac{dy}{dz}\right)$$

$$=\frac{a}{ax+b}\cdot\frac{d}{dx}\left(\frac{dy}{dz}\right)+\frac{dy}{dz}\cdot\frac{d}{dx}\left(\frac{a}{ax+b}\right)$$

$$=\frac{a}{ax+b}\cdot\left[\frac{d}{dz}\left(\frac{dy}{dz}\right)\cdot\frac{dz}{dx}\right]+\frac{dy}{dz}\cdot\left(-\frac{a^2}{(ax+b)^2}\right)$$

$$=\frac{a}{ax+b}\cdot\left[\frac{d^2y}{dz^2}\cdot\frac{a}{ax+b}\right]-\frac{a^2}{(ax+b)^2}\frac{dy}{dz}$$

$$\therefore \frac{d^2y}{dx^2}=\frac{a^2}{(ax+b)^2}\frac{d^2y}{dz^2}-\frac{a^2}{(ax+b)^2}\frac{dy}{dz}$$

Multiplying by $(ax+b)^2$

$$(ax+b)^2\frac{d^2y}{dx^2}=a^2\frac{d^2y}{dz^2}-a^2\frac{dy}{dz}=a^2\left(D^2y-Dy\right)$$

$$\therefore\left(ax+b\right)^2\frac{d^2y}{dx^2}=a^2D\left(D-1\right)y$$

Similarly, $\left(ax+b\right)^3\frac{d^3y}{dx^3}=a^3D\left(D-1\right)\left(D-2\right)y$

Substituting $\left(ax+b\right)\frac{dy}{dx}=Dy,\ \left(ax+b\right)^2\frac{d^2y}{dx^2}=a^2D\left(D-1\right)y,$

$\left(ax+b\right)^3\frac{d^3y}{dx^3}=a^3D\left(D-1\right)\left(D-2\right)y$ in Legendre's differential equation, we get a linear differential equation with constant coefficients, which can be solved using methods discussed in previous sections.

(7.9) Solve following differential equations

1. $(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=4\cos[\log(1+x)]$

2. $(x-1)^3\frac{d^3y}{dx^3}+2(x-1)^2\frac{d^2y}{dx^2}-4(x-1)\frac{dy}{dx}+4y=4\log(x-1)$

3. $(3x+2)^2\frac{d^2y}{dx^2}+3(3x+2)\frac{dy}{dx}-36y=3x^2+4x+1$

4. $(x+2)^2\frac{d^2y}{dx^2}-(x+2)\frac{dy}{dx}+y=3x+4$

5. Solve $(x+1)^2\frac{d^2y}{dx^2}+(x+1)\frac{dy}{dx}=(2x+3)(2x+4)$

Example 7.9.1: Solve: $(1+x)^2 \frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=4\cos[\log(1+x)]$

Solution: Given differential equation is Legendre's Differential Equation. To convert it into linear differential equation with constant coefficients, we put $(1+x)=e^z$

$$\therefore z=\log(1+x) \text{ and } \frac{dz}{dx}=\frac{1}{1+x}$$

By chain rule, $\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{dy}{dz}\cdot\frac{1}{1+x}$ $\quad\therefore(1+x)\frac{dy}{dx}=\frac{dy}{dz}=Dy,$ where $D=\frac{d}{dz}$

Similarly, $(1+x)^2\frac{d^2y}{dx^2}=D(D-1)y$

Thus, given differential equation becomes $D(D-1)y+Dy+y=4\cos z$

$\therefore\left(D^2-D+D+1\right)y=4\cos z$

$\therefore\left(D^2+1\right)y=4\cos z$

Auxiliary Equation is $m^2+1=0$

$$\therefore m=\pm\sqrt{-1}=\pm i \qquad \{\text{Roots are complex conjugate}\}$$

$\therefore$ Complementary Function $=e^{0z}\left[c_1\cos z+c_2\sin z\right]=c_1\cos\left[\log(1+x)\right]+c_2\sin\left[\log(1+x)\right]$

$P.I.=\frac{1}{D^2+1}4\cos z$

$=4\frac{z}{2D}\cos z \qquad \{\because$ Denominator becomes zero, when D^2 is replaced by $-1^2=-1\}$

$=2z\frac{1}{D}\cos z=2z\int\cos z\,dz=2z\sin z$

$\therefore$ Particular Integral $=2\log(1+x)\cdot\sin\left[\log(1+x)\right]$

General Solution = Complementary Function + Particular Integral

$\therefore \boxed{y=c_1\cos\left[\log(1+x)\right]+c_2\sin\left[\log(1+x)\right]+2\log(1+x)\cdot\sin\left[\log(1+x)\right]}$

Example 7.9.2: Solve: $(x-1)^3\dfrac{d^3y}{dx^3}+2(x-1)^2\dfrac{d^2y}{dx^2}-4(x-1)\dfrac{dy}{dx}+4y=4\log(x-1)$

Solution: Given differential equation is Legendre's Differential Equation. To convert it into linear differential equation with constant coefficients, we put $(x-1)=e^z$

$\therefore z=\log(x-1)$ and $\dfrac{dz}{dx}=\dfrac{1}{x-1}$

By chain rule, $\dfrac{dy}{dx}=\dfrac{dy}{dz}\cdot\dfrac{dz}{dx}=\dfrac{dy}{dz}\cdot\dfrac{1}{x-1}$

$\therefore (x-1)\dfrac{dy}{dx}=\dfrac{dy}{dz}=Dy,$ where $D=\dfrac{d}{dz}$

Similarly, $(x-1)^2\dfrac{d^2y}{dx^2}=D(D-1)y$ and $(x-1)^3\dfrac{d^3y}{dx^3}=D(D-1)(D-2)y$

Thus, given differential equation becomes

$\therefore\ D(D-1)(D-2)y+2\,D(D-1)y-4Dy+4y=4z$

$\therefore\left(D^3-3D^2+2D+2D^2-2D-4D+4\right)y=4z$

$\therefore\left(D^3-D^2-4D+4\right)y=4\cos z$

Auxiliary Equation is $m^3-m^2-4m+4=0$

$\therefore m^2(m-1)-4(m-1)=0$

$\therefore (m-1)\left(m^2-4\right)=0$

$\therefore m=1,2,-2\quad$ {Roots are real and distinct}

$\therefore$ Complementary Function $=c_1e^z+c_2e^{2z}+c_3e^{-2z}=c_1(x-1)+c_2(x-1)^2+c_3\dfrac{1}{(x-1)^2}$

$$P.I.=\frac{1}{D^3-D^2-4D+4}4z$$

$$=\frac{1}{4\left(1+\dfrac{D^3-D^2-4D}{4}\right)}4z$$

$$P.I = \frac{1}{\left(1+\frac{D^3-D^2-4D}{4}\right)} z$$

$$= \left[1-\left(\frac{D^3-D^2-4D}{4}\right)+\cdots\right] z$$

$$= \left[1+D+\cdots\right] z$$

$$= \left[z+Dz+\cdots\right] = z+1$$

$\therefore$ Particular Integral $= \log(x-1)+1$

General Solution = Complementary Function + Particular Integral

$$\therefore \boxed{y = c_1(x-1)+c_2(x-1)^2+c_3\frac{1}{(x-1)^2}+\log(x-1)+1}$$

Example 7.9.3: Solve $(3x+2)^2\frac{d^2y}{dx^2}+3(3x+2)\frac{dy}{dx}-36y=3x^2+4x+1$

Solution: Given differential equation is Legendre's Differential Equation. To convert it into linear differential equation with constant coefficients, we put $(3x+2)=e^z$

$\therefore z=\log(3x+2)$ and $\frac{dz}{dx}=\frac{3}{3x+2}$

By chain rule, $\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{dy}{dz}\cdot\frac{3}{3x+2}$

$\therefore(3x+2)\frac{dy}{dx}=3\frac{dy}{dz}=3Dy,\quad$ where $D=\frac{d}{dz}$

Similarly, $(3x+2)^2\frac{d^2y}{dx^2}=3^2D(D-1)y=(9D^2-9D)y$

Thus, given differential equation becomes

$$(9D^2-9D)y+3(3\,Dy)-36y=3\left(\frac{e^z-2}{3}\right)^2+4\left(\frac{e^z-2}{3}\right)+1$$

$$\therefore(9D^2-9D+9D-36)y=3\frac{e^{2z}-4e^z+4}{9}+\frac{4e^z-8}{3}+1$$

$$\therefore 9(D^2-4)y=\frac{e^{2z}-4e^z+4+4e^z-8+3}{3}$$

$$\therefore 9(D^2-4)y=\frac{e^{2z}-1}{3}$$

$$\therefore(D^2-4)y=\frac{e^{2z}-1}{27}$$

Auxiliary Equation is $m^2-4=0$

$\therefore(m-2)(m+2)=0$

$\therefore m=2,-2\quad$ {Roots are real and distinct}

$\therefore$ Complementary Function $=c_1e^{2z}+c_2e^{-2z}=c_1(3x+2)^2+c_2\frac{1}{(3x+2)^2}$

$$P.I = \frac{1}{D^2-4}\left(\frac{e^{2z}-1}{27}\right)$$

$$= \frac{1}{27}\left\{\frac{1}{D^2-4}e^{2z} - \frac{1}{D^2-4}e^{0z}\right\} \qquad \text{Putting } 1 = e^{0z}$$

$$= \frac{1}{27}\left\{\frac{z}{2D}e^{2z} - \frac{1}{0-4}e^{0z}\right\}$$

$$= \frac{1}{27}\left\{\frac{z}{2}\int e^{2z}dz + \frac{1}{4}\right\}$$

$$= \frac{1}{27}\left\{\frac{z}{4}e^{2z} + \frac{1}{4}\right\}$$

$$= \frac{1}{108}\left(ze^{2z}+1\right)$$

$\therefore$ Particular Integral $= \frac{1}{108}\left[(3x+2)^2 \cdot \log(3x+2)+1\right]$

General Solution = Complementary Function + Particular Integral

$$\therefore \boxed{y = c_1(3x+2)^2 + c_2\frac{1}{(3x+2)^2} + \frac{1}{108}\left[(3x+2)^2 \cdot \log(3x+2)+1\right]}$$

Example 7.9.4: Solve $(x+2)^2\frac{d^2y}{dx^2}-(x+2)\frac{dy}{dx}+y=3x+4$

Solution: Given differential equation is Legendre's Differential Equation. To convert it into linear differential equation with constant coefficients, we put $(x+2)=e^z$

$\therefore z=\log(x+2)$ and $\frac{dz}{dx}=\frac{1}{x+2}$

By chain rule $\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{dy}{dz}\cdot\frac{1}{x+2}$ $\quad\therefore(x+2)\frac{dy}{dx}=\frac{dy}{dz}=Dy,\quad$ where $D=\frac{d}{dz}$

Similarly, $(x+2)^2\frac{d^2y}{dx^2}=D(D-1)y=(D^2-D)y$

Thus, given differential equation becomes $(D^2-D)y-Dy+y=3(e^z-2)+4$

$\therefore(D^2-D-D+1)y=3e^z-6+4$

$\therefore(D^2-2D+1)y=3e^z-2$

Auxiliary Equation is $m^2-2m+1=0$

$\therefore(m-1)^2=0$

$\therefore m=1,1\quad$ {Roots are real and equal}

$\therefore$ Complementary Function = $(c_1+c_2z)e^z=[c_1+c_2\log(x+2)]\cdot(x+2)$

$$P.I=\frac{1}{D^2-2D+1}(3e^z-2)\;=\;3\frac{1}{D^2-2D+1}e^z\;-\;2\frac{1}{D^2-2D+1}e^{0z}$$

$$=3\frac{z}{2D-2}e^z-2\frac{1}{0-0+1}e^{0z}$$

$$=3\frac{z^2}{2}e^z-2$$

Particular Integral $=\frac{3}{2}(x+2)[\log(x+2)]^2-2$

General Solution = Complementary Function + Particular Integral

$$\therefore \boxed{y=[c_1+c_2\log(x+2)]\cdot(x+2)+\frac{3}{2}(x+2)[\log(x+2)]^2-2}$$

Example 7.9.5: Solve Solve $(x+1)^2\dfrac{d^2y}{dx^2}+(x+1)\dfrac{dy}{dx}=(2x+3)(2x+4)$

Solution: Given differential equation is Legendre's Differential Equation. To convert it into linear differential equation with constant coefficients, we put $(x+1)=e^z$

$\therefore z=\log(x+1)$ and $\dfrac{dz}{dx}=\dfrac{1}{x+1}$

By chain rule $\dfrac{dy}{dx}=\dfrac{dy}{dz}\cdot\dfrac{dz}{dx}=\dfrac{dy}{dz}\cdot\dfrac{1}{x+1}$ $\quad\therefore(x+1)\dfrac{dy}{dx}=\dfrac{dy}{dz}=Dy,$ where $D=\dfrac{d}{dz}$

Similarly, $(x+1)^2\dfrac{d^2y}{dx^2}=D(D-1)y=(D^2-D)y$

Thus, given differential equation becomes $(D^2-D)y+Dy=\left[2(e^z-1)+3\right]\left[2(e^z-1)+4\right]$

$\therefore(D^2-D+D)y=(2e^z+1)(2e^z+2)$

$\therefore D^2y=4e^{2z}+6e^z+2$

Auxiliary Equation is $m^2=0$

$\therefore m=0,0\quad$ {Roots are real and equal}

$\therefore$ Complementary Function = $(c_1+c_2z)e^{0z}=\left[c_1+c_2\log(x+1)\right]$

$$P.I=\frac{1}{D^2}\left(4e^{2z}+6e^z+2\right)=4\frac{1}{D^2}e^{2z}+6\frac{1}{D^2}e^z+\frac{1}{D^2}2$$

$$=4\frac{1}{2^2}e^{2z}+6\frac{1}{1^2}e^z+\frac{1}{D}\left[\frac{1}{D}2\right]$$

$$=e^{2z}+6e^z+z^2\qquad\left\{\because\frac{1}{D}\left[\frac{1}{D}2\right]=\frac{1}{D}\left[\int 2dz\right]=\frac{1}{D}2z=\int 2zdz=z^2\right\}$$

Particular Integral $=(x+1)^2+6(x+1)+\left[\log(x+1)\right]^2$

General Solution = Complementary Function + Particular Integral

$$\therefore\boxed{y=\left[c_1+c_2\log(x+1)\right]+(x+1)^2+6(x+1)+\left[\log(x+1)\right]^2}$$