Chapter-5	First Order and First Degree
Topic-5.5	Ordinary Differential Equations

Type 5: - First Order Linear Differential Equation

Most General form of first order linear differential equation is

$$\frac{dy}{dx}+P(x)y=Q(x)$$

wher $P(x)$ and $Q(x)$ are functions of x

Observe that

- **(i) There are 3 terms (One term on RHS and Two terms on LHS)**
- **(ii) RHS is function of independent variable (Here x is the independent variable and y is the dependent variable)**
- **(iii) Coefficient of the derivative $\frac{dy}{dx}$ is 1(one).**

Integrating Factor (I.F) is $\boxed{e^{\int P(x)dx}}$

and Solution is $\boxed{y\cdot(I.F)=\int(I.F)Qdx+C}$, where C is an arbitrary constant.

Differential Equation $\frac{dx}{dy}+P(y)x=Q(y)$ is first order linear differential equation.

Its Integrating factor is $e^{\int P(y)dy}$ and Solution is $x\cdot(I.F)=\int(I.F)Qdy+C$

Differential Equation $\frac{dv}{dt}+P(t)v=Q(t)$ is also first order linear differential equation.

Its Integrating factor is $e^{\int P(t)dt}$ and Solution is $v\cdot(I.F)=\int(I.F)Qdt+C$

(5.5) Solve following differential equations

1. $\frac{dy}{dx}+\frac{y}{x}=x^2$

2. $\frac{dy}{dx}-y\tan x=1$

3. $\frac{dy}{dx}-\frac{3y}{x}+x^3-x=0$

4. $\frac{dy}{dx}+3x^2y=6x^2$

5. $\frac{dy}{dx}+\left(\frac{1+x}{x}\right)y=\frac{e^x}{x}$

6. $\frac{dy}{dx}-\frac{2y}{x}=\frac{5x^2}{(2+x)(3-2x)}$

7. $\frac{dy}{dx}+\frac{4x}{x^2+1}y=\frac{1}{(x^2+1)^3}$

8. $\frac{dx}{dy}-5x=\sin y$

9. $x\frac{dy}{dx}+y=\sqrt{x}$

10. $4x^3y+x^4\frac{dy}{dx}=\sin^3 x$

11. $(1-x^2)\frac{dy}{dx}-xy=1$

12. $x\log x\frac{dy}{dx}+y=2\log x$

13. $\cos y\frac{dx}{dy}+x\sin y=\sec^2 y$

14. $\operatorname{cosec} x\frac{dy}{dx}=y+\cos x$

15. $x(x-1)\frac{dy}{dx}-(x-2)y=x^3(2x-1)$

Example 5.5.1: Solve $\frac{dy}{dx}+\frac{y}{x}=x^2$

Solution: Writing given differential equation as $\frac{dy}{dx}+\left(\frac{1}{x}\right)y=x^2$

This is a first order linear differential equation of the form $\frac{dy}{dx}+P(x)y=Q(x)$

where, $P(x)=\frac{1}{x}$ and $Q(x)=x^2$

$$\int P(x)\,dx=\int\frac{1}{x}dx=\log x$$

$$\therefore I.F=e^{\int P(x)dx}=e^{\log x}=x$$

Its solution is $y\cdot(I.F)=\int(I.F)\cdot Q(x)\,dx+C$

$$\therefore y\cdot x=\int x\cdot x^2 dx+C$$

$$\therefore yx=\int x^3 dx+C$$

$$\therefore \boxed{yx=\frac{x^4}{4}+C}$$

Example 5.5.2: Solve $\frac{dy}{dx}-y\tan x=1$

Solution: Writing given differential equation as $\frac{dy}{dx}+(-\tan x)y=1$

This is a first order linear differential equation of the form $\frac{dy}{dx}+P(x)y=Q(x)$

where, $P(x)=-\tan x$ and $Q(x)=1$

$\int P(x)dx=\int -\tan x dx=-\log\sec x=\log(\sec x)^{-1}$

$\therefore I.F=e^{\int P(x)dx}=e^{\log(\sec x)^{-1}}=(\sec x)^{-1}=\frac{1}{\sec x}=\cos x$

Its solution is $y\cdot(I.F)=\int(I.F)\cdot Q(x)dx+C$

$\therefore y\cdot\cos x=\int\cos x\cdot(1)dx+C$

$\therefore y\cos x=\sin x+C$

$\therefore \boxed{y=\tan x+C\cdot\sec x}$

Example 5.5.3: Solve $\frac{dy}{dx}-\frac{3y}{x}+x^3-x=0$

Solution: Writing given differential equation as $\frac{dy}{dx}+\left(-\frac{3}{x}\right)y=x-x^3$

This is a first order linear differential equation of the form $\frac{dy}{dx}+P(x)y=Q(x)$

where, $P(x)=\frac{-3}{x}$ and $Q(x)=x-x^3$

$$\int P(x)\,dx=\int\frac{-3}{x}dx=-3\log x=\log x^{-3}$$

$$\therefore I.F=e^{\int P(x)dx}=e^{\log x^{-3}}=x^{-3}=\frac{1}{x^3}$$

Its solution is $y\cdot(I.F)=\int(I.F)\cdot Q(x)\,dx+C$

$$\therefore y\cdot\frac{1}{x^3}=\int\frac{1}{x^3}\cdot\left(x-x^3\right)dx+C$$

$$\therefore \frac{y}{x^3}=\int\left(\frac{1}{x^2}-1\right)dx+C$$

$$\therefore \frac{y}{x^3}=\left[-\frac{1}{x}-1\right]+C$$

$$\therefore \boxed{\frac{y}{x^3}=-\left[\frac{1}{x}+1\right]+C} \text{ or } \boxed{y=-x^2(1+x)+Cx^3}$$

Example 5.5.4: Solve $\frac{dy}{dx}+3x^2y=6x^2$

Solution: Writing given differential equation as $\frac{dy}{dx}+\left(3x^2\right)y=6x^2$

This is a first order linear differential equation of the form $\frac{dy}{dx}+P\left(x\right)y=Q\left(x\right)$

where, $P\left(x\right)=3x^2$ and $Q\left(x\right)=6x^2$

$\int P\left(x\right)dx=\int 3x^2dx=3\frac{x^3}{3}=x^3$

$\therefore I.F=e^{\int P(x)dx}=e^{x^3}$

Its solution is $y\cdot\left(I.F\right)=\int\left(I.F\right)\cdot Q\left(x\right)dx+C$

$\therefore y\cdot e^{x^3}=\int e^{x^3}\cdot\left(6x^2\right)dx+C$

$\therefore y\cdot e^{x^3}=\int 2e^{x^3}\cdot\left(3x^2\right)dx+C$

On RHS, putting $x^3=t,\quad \therefore 3x^2dx=dt$

$\therefore y\cdot e^{x^3}=2\int e^t dt+C$

$y\cdot e^{x^3}=2e^t+C$

$y\cdot e^{x^3}=2e^{x^3}+C\quad \left\{\because t=x^3\right\}$

$\therefore \boxed{y=2+Ce^{-x^3}}$

Example 5.5.5: Solve $\frac{dy}{dx}+\left(\frac{1+x}{x}\right)y=\frac{e^x}{x}$

Solution: $\frac{dy}{dx}+\left(\frac{1+x}{x}\right)y=\frac{e^x}{x}$

This is a first order linear differential equation of the form $\frac{dy}{dx}+P(x)y=Q(x)$

where, $P(x)=\frac{1+x}{x}$ and $Q(x)=\frac{e^x}{x}$

$$\int P(x)dx=\int\left(\frac{1+x}{x}\right)dx=\int\left(\frac{1}{x}+1\right)dx=\log x+x$$

$$\therefore I.F=e^{\int P(x)dx}=e^{\log x+x}=e^{\log x}\cdot e^x=xe^x$$

Its solution is $y\cdot(I.F)=\int(I.F)\cdot Q(x)dx+C$

$$\therefore y\cdot\left(xe^x\right)=\int xe^x\cdot\left(\frac{e^x}{x}\right)dx+C$$

$$\therefore yxe^x=\int e^{2x}dx+C$$

$$yxe^x=\frac{e^{2x}}{2}+C$$

$$\therefore \boxed{2yxe^x=e^{2x}+2C}$$

Example 5.5.6: Solve $\frac{dy}{dx}-\frac{2y}{x}=\frac{5x^2}{(2+x)(3-2x)}$

Solution: Writing given differential equation as $\frac{dy}{dx}+\left(-\frac{2}{x}\right)y=\frac{5x^2}{(2+x)(3-2x)}$

This is a first order linear differential equation of the form $\frac{dy}{dx}+P(x)y=Q(x)$

where, $P(x)=-\frac{2}{x}$ and $Q(x)=\frac{5x^2}{(2+x)(3-2x)}$

$\int P(x)\,dx=\int -\frac{2}{x}\,dx=-2\log x=\log x^{-2}$ $\quad\therefore I.F=e^{\int P(x)dx}=e^{\log x^{-2}}=x^{-2}=\frac{1}{x^2}$

Its solution is $y\cdot(I.F)=\int(I.F)\cdot Q(x)\,dx+C$

$$\therefore y\cdot\frac{1}{x^2}=\int\frac{1}{x^2}\frac{5x^2}{(2+x)(3-2x)}dx+C=\int\frac{5}{(2+x)(3-2x)}dx+C------(1)$$

Consider $\frac{5}{(2+x)(3-2x)}=\frac{A}{2+x}+\frac{B}{3-2x}$ $\{$Resolving into partial fraction$\}$

$\therefore A(3-2x)+B(2+x)=5$

Putting $x=-2,\ A(3+4)+B(0)=5\quad\therefore A=\frac{5}{7}$

Putting $x=\frac{3}{2},\ A(0)+B\left(2+\frac{3}{2}\right)=5\quad\therefore B=\frac{10}{7}$

$$\frac{5}{(2+x)(3-2x)}=\frac{5/7}{2+x}+\frac{10/7}{3-2x}----------(2)$$

$\therefore$ From (1) and (2), Solution is $\frac{y}{x^2}=\int\left[\frac{5/7}{2+x}+\frac{10/7}{3-2x}\right]dx+C$

$$\therefore\frac{y}{x^2}=\frac{5}{7}\log(2+x)+\frac{10}{7}\left[\frac{\log(3-2x)}{-2}\right]$$

$$\therefore\boxed{\frac{y}{x^2}=\frac{5}{7}\log\left(\frac{2+x}{3-2x}\right)+C}$$

Example 5.5.7: Solve $\frac{dy}{dx}+\frac{4x}{x^2+1}y=\frac{1}{\left(x^2+1\right)^3}$

Solution: Writing given differential equation as $\frac{dy}{dx}+\left(\frac{4x}{x^2+1}\right)y=\frac{1}{\left(x^2+1\right)^3}$

This is a first order linear differential equation of the form $\frac{dy}{dx}+P(x)y=Q(x)$

where, $P(x)=\frac{4x}{x^2+1}$ and $Q(x)=\frac{1}{\left(x^2+1\right)^3}$

$$\int P(x)\,dx=\int\frac{4x}{x^2+1}dx=2\int\frac{2x}{x^2+1}dx=2\log\left(x^2+1\right)=\log\left(x^2+1\right)^2$$

$$\therefore I.F=e^{\int P(x)dx}=e^{\log\left(x^2+1\right)^2}=\left(x^2+1\right)^2$$

Its solution is $y\cdot(I.F)=\int(I.F)\cdot Q(x)\,dx+C$

$$\therefore y\cdot\left(x^2+1\right)^2=\int\left(x^2+1\right)^2\cdot\frac{1}{\left(x^2+1\right)^3}dx+C$$

$$\therefore y\cdot\left(x^2+1\right)^2=\int\frac{1}{\left(x^2+1\right)}dx+C\text{------}(1)$$

$$\therefore \boxed{y\cdot\left(x^2+1\right)^2=\tan^{-1}x+C}$$

Example 5.5.8: Solve $\frac{dx}{dy} - 5x = \sin y$

Solution: Writing given differential equation as $\frac{dx}{dy} + (-5)x = \sin y$

This is a first order linear differential equation of the form $\frac{dx}{dy} + P(y)x = Q(y)$

where, $P(y) = -5$ and $Q(y) = \sin y$

$$\int P(y)\,dy = \int -5dy = -5y$$

$$\therefore I.F = e^{\int P(y)dy} = e^{-5y}$$

Its solution is $x \cdot (I.F) = \int (I.F) \cdot Q(y)\,dy + C$

$$\therefore x \cdot e^{-5y} = \int e^{-5y} \cdot (\sin y)dy + C$$

$$xe^{-5y} = \frac{e^{-5y}}{(-5)^2 + 1^2}\left[-5\sin y - \cos y\right] + C \quad \left\{ \text{Using } \int e^{ax}(\sin bx)\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) \right\}$$

$$xe^{-5y} = -\frac{e^{-5y}}{26}\left[5\sin y + \cos y\right] + C$$

$$\therefore \boxed{x = -\frac{1}{26}\left[5\sin y + \cos y\right] + ce^{5y}}$$

Example 5.5.9: Solve $x\dfrac{dy}{dx}+y=\sqrt{x}$

Solution: Dividing both the sides of differential equation by x

$$\frac{dy}{dx}+\frac{1}{x}y=\frac{1}{\sqrt{x}}$$

This is a first order linear differential equation of the form $\dfrac{dy}{dx}+P(x)y=Q(x)$

where, $P(x)=\dfrac{1}{x}$ and $Q(x)=\dfrac{1}{\sqrt{x}}$

$$\int P(x)\,dx=\int\frac{1}{x}dx=\log x$$

$$\therefore I.F=e^{\int P(x)dx}=e^{\log x}=x$$

Its solution is $y\cdot(I.F)=\int(I.F)\cdot Q(x)\,dx+C$

$$\therefore y\cdot x=\int x\cdot\frac{1}{\sqrt{x}}dx+C$$

$$\therefore xy=\int\sqrt{x}\,dx+C$$

$$\therefore xy=\frac{x^{3/2}}{3/2}+C$$

$$\therefore \boxed{xy=\frac{2}{3}x^{3/2}+C}$$

Example 5.5.10: Solve $4x^3y+x^4\dfrac{dy}{dx}=\sin^3 x$

Solution: Writing given differential equation as $x^4\dfrac{dy}{dx}+4x^3y=\sin^3 x$

Dividing both the sides of differential equation by x^4

$$\frac{dy}{dx}+\frac{4}{x}y=\frac{\sin^3 x}{x^4}$$

This is a first order linear differential equation of the form $\dfrac{dy}{dx}+P(x)y=Q(x)$

where, $P(x)=\dfrac{4}{x}$ and $Q(x)=\dfrac{\sin^3 x}{x^4}$

$$\int P(x)dx=\int\frac{4}{x}dx=4\log x=\log x^4$$

$$\therefore I.F=e^{\int P(x)dx}=e^{\log x^4}=x^4$$

Its solution is $y\cdot(I.F)=\int(I.F)\cdot Q(x)dx+C$

$$\therefore y\cdot x^4=\int x^4\cdot\frac{\sin^3 x}{x^4}dx+C$$

$$\therefore yx^4=\int\sin^3 x dx+C$$

$$\therefore yx^4=\int\left[\frac{3\sin x-\sin 3x}{4}\right]dx+C$$

$$\therefore yx^4=\frac{3}{4}(-\cos x)-\frac{1}{4}\left(-\frac{\cos 3x}{3}\right)+C$$

$$\therefore \boxed{yx^4=-\frac{3}{4}\cos x+\frac{1}{12}\cos 3x+C}$$

Example 5.5.11: Solve $\left(1-x^2\right)\frac{dy}{dx}-xy=1$

Solution: Dividing both the sides of differential equation by $1-x^2$

$$\frac{dy}{dx}+\left(-\frac{x}{1-x^2}\right)y=\frac{1}{1-x^2}$$

This is a first order linear differential equation of the form $\frac{dy}{dx}+P(x)y=Q(x)$

where, $P(x)=-\frac{x}{1-x^2}$ and $Q(x)=\frac{1}{1-x^2}$

$$\int P(x)\,dx=\int -\frac{x}{1-x^2}\,dx=\frac{1}{2}\int\frac{-2x}{1-x^2}\,dx=\frac{1}{2}\log\left(1-x^2\right)=\log\sqrt{1-x^2}$$

$$\therefore I.F=e^{\int P(x)dx}=e^{\log\sqrt{1-x^2}}=\sqrt{1-x^2}$$

Its solution is $y\cdot(I.F)=\int(I.F)\cdot Q(x)\,dx+C$

$$\therefore y\sqrt{1-x^2}=\int\sqrt{1-x^2}\cdot\frac{1}{1-x^2}\,dx+C$$

$$\therefore y\sqrt{1-x^2}=\int\frac{1}{\sqrt{1-x^2}}\,dx+C$$

$$\therefore \boxed{y\sqrt{1-x^2}=\sin^{-1}x+C}$$

Example 5.5.12: Solve $x\log x\dfrac{dy}{dx}+y=2\log x$

Solution: Dividing both the sides of differential equation by $x\log x$

$$\frac{dy}{dx}+\frac{1}{x\log x}y=\frac{2}{x}$$

This is a first order linear differential equation of the form $\dfrac{dy}{dx}+P(x)y=Q(x)$

where, $P(x)=\dfrac{1}{x\log x}$ and $Q(x)=\dfrac{2}{x}$

$$\int P(x)dx=\int\frac{1}{x\log x}dx=\int\frac{1}{\log x}\left(\frac{1}{x}dx\right)$$

Put $\log x=t \quad \therefore \dfrac{1}{x}dx=dt$

$$\therefore \int P(x)=\int\frac{1}{\log x}\left(\frac{1}{x}dx\right)=\int\frac{1}{t}dt=\log t=\log(\log x)$$

$$\therefore I.F=e^{\int P(x)dx}=e^{\log(\log x)}=\log x$$

Its solution is $y\cdot(I.F)=\int(I.F)\cdot Q(x)dx+C$

$$\therefore y\cdot\log x=\int\log x\cdot\frac{1}{x}dx+C$$

On RHS, put $\log x=t \quad \therefore \dfrac{1}{x}dx=dt$

$$\therefore y\cdot\log x=\int t\,dt+C$$

$$\therefore y\cdot\log x=\frac{t^2}{2}+C$$

$$\therefore \boxed{y\log x=\frac{(\log x)^2}{2}+C}$$

Example 5.5.13: Solve $\cos y \frac{dx}{dy} + x \sin y = \sec^2 y$

Solution: Dividing both the sides of differential equation by $\cos y$

$$\frac{dx}{dy} + (\tan y)x = \sec^3 y$$

This is a first order linear differential equation of the form $\frac{dx}{dy} + P(y)x = Q(y)$

where, $P(y) = \tan y$ and $Q(y) = \sec^3 y$

$$\int P(y)\,dy = \int \tan y\,dy = \log \sec y$$

$$\therefore I.F = e^{\int P(y)dy} = e^{\log \sec y} = \sec y$$

Its solution is $x \cdot (I.F) = \int (I.F) \cdot Q(y)\,dy + C$

$$\therefore x \cdot \sec y = \int \sec y \cdot (\sec^3 y) dy + C$$

$$x \sec y = \int \sec^2 y \cdot \sec^2 y\,dy + C$$

$$x \sec y = \int (1 + \tan^2 y) \cdot \sec^2 y\,dy + C$$

On RHS, Put $\tan y = t \quad \therefore \sec^2 y\,dy = dt$

$$x \sec y = \int (1 + t^2)\,dt + C$$

$$x \sec y = t + \frac{t^3}{3} + C$$

$$\therefore \boxed{x \sec y = \tan y + \frac{\tan^3 y}{3} + C}$$

Example 5.5.14: Solve $\cos ec x \frac{dy}{dx} = y + \cos x$

Solution: Dividing both the sides of differential equation by $\cos ec x$

$$\frac{dy}{dx} - \frac{1}{\cos ec x} y = \frac{\cos x}{\cos ec x}$$

$$\therefore \frac{dy}{dx} + (-\sin x) y = \sin x \cdot \cos x$$

This is a first order linear differential equation of the form $\frac{dy}{dx} + P(x) y = Q(x)$

where, $P(x) = -\sin x$ and $Q(x) = \sin x \cdot \cos x$

$$\int P(x) dx = \int -\sin x dx = \cos x$$

$$\therefore I.F = e^{\int P(x) dx} = e^{\cos x}$$

Its solution is $y \cdot (I.F) = \int (I.F) \cdot Q(x) dx + C$

$$\therefore y\left(e^{\cos x}\right) = \int e^{\cos x} \cdot \sin x \cdot \cos x dx + C$$

$$\therefore y\left(e^{\cos x}\right) = \int e^{\cos x} \cdot \cos x \cdot \sin x dx + C$$

On RHS, put $\cos x = t \quad \therefore -\sin x dx = dt$

$$\therefore y e^{\cos x} = \int e^{t} \cdot t(-dt) + C$$

$$\therefore y e^{\cos x} = -\int t \cdot e^{t} dt + C$$

$$\therefore y e^{\cos x} = -\left[t \cdot e^{t} - (1) \cdot e^{t}\right] + C \quad \left\{\text{Integrating by parts using } \int uv dx = uv_1 - u'v_2 + u''v_3 \cdots\right\}$$

$$\therefore y e^{\cos x} = -e^{\cos x}\left[\cos x - 1\right] + C \quad \{\because t = \cos x\}$$

$$\therefore \boxed{y = -(\cos x - 1) + Ce^{-\cos x}}$$

Example 5.5.15: $x(x-1)\frac{dy}{dx}-(x-2)y=x^3(2x-1)$

Solution: Dividing both the sides of differential equation by $x(x-1)$

$$\therefore \frac{dy}{dx}-\frac{x-2}{x(x-1)}y=\frac{x^2(2x-1)}{x-1}$$

This is a first order linear differential equation of the form $\frac{dy}{dx}+P(x)y=Q(x)$

where, $P(x)=-\left[\frac{x-2}{x(x-1)}\right]$ and $Q(x)=\frac{x^2(2x-1)}{x-1}$

$$\int P(x)dx=-\int\frac{x-2}{x(x-1)}dx=-\int\frac{2(x-1)-x}{x(x-1)}dx=-\int\left[\frac{2}{x}-\frac{1}{x-1}\right]dx$$

$$=-2\log x+\log(x-1)=\log\left(\frac{x-1}{x^2}\right)$$

$$\therefore I.F=e^{\int P(x)dx}=e^{\log\left(\frac{x-1}{x^2}\right)}=\left(\frac{x-1}{x^2}\right)$$

Its solution is $y\cdot(I.F)=\int(I.F)\cdot Q(x)dx+C$

$$\therefore y\left(\frac{x-1}{x^2}\right)=\int\left(\frac{x-1}{x^2}\right)\cdot\left(\frac{x^2(2x-1)}{x-1}\right)dx+C$$

$$\therefore y\left(\frac{x-1}{x^2}\right)=\int(2x-1)dx+C$$

$$\therefore \boxed{y\left(\frac{x-1}{x^2}\right)=x^2-x+C}$$