Chapter-1	Double Integrals
Topic-1.1	

What is a double integral?

Recall that a single integral is something of the form

$$\int_a^b f(x)\,dx$$

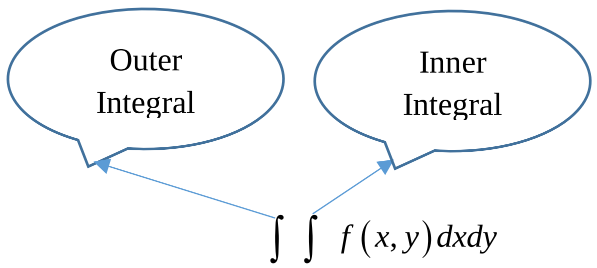
A double integral is something of the form

$$\iint_R f(x,y)\,dxdy$$

where R is called the region of integration and is a region in the XY plane.

We can evaluate double integrals in two steps:

First evaluate the inner integral, and then plug this solution into the outer integral and solve that.



Note:

1. **Limits of Outer Integral are always constant**

2. **Limits of Inner Integral are constant or function of only one variable (either *x* or *y*)**

Working Rule to Evaluate Double Integrals

Case 1: Limits of Inner Integral are function of x. That is,

$$\int_a^b \int_{h_1(x)}^{h_2(x)} f(x,y)\,dxdy$$

Step (i): First integrate $f(x,y)$ with respect to y (keeping x constant)

Step (ii) : Integrate the result obtained in step (i) with respect to x

Here order of integration is first w.r.t y and then w.r.t. x (YX order)

Note that order of integration is decided by the limits of inner integral and not by $dxdy$ or $dydx$

For Example: Evaluate $\int_0^2 \int_0^{x^2} xydxdy$

Solution: Note that upper limit of inner integral is function of x. Therefore, order of integration is first w.r.t y and then w.r.t. x.

$$I = \int_{x=0}^{x=2} \int_{y=0}^{y=x^2} xydxdy$$

Place the inner integral in parentheses so you can better see what you're working with:

$$I = \int_{x=0}^{x=2} \left(\int_{y=0}^{y=x^2} xydy \right) dx$$

Now focus on what's inside the parentheses. For the moment, you can ignore the rest. Your integration variable is *y*, so treat the variable *x* as a constant, moving it outside the inner integral:

$$I = \int_{x=0}^{x=2} x \left(\int_{y=0}^{y=x^2} ydy \right) dx$$

$$= \int_{x=0}^{x=2} x \left(\frac{y^2}{2} \right)_0^{x^2} dx$$

$$= \frac{1}{2}\int_{x=0}^{x=2} x\left(x^4 - 0\right)dx = \frac{1}{2}\int_{x=0}^{x=2} x^5 dx$$

$$= \frac{1}{2}\left[\frac{x^6}{6}\right]_0^2 = \frac{1}{12}\left[2^6 - 0\right] = \frac{64}{12} = \frac{16}{3}$$

Case II: Limits of Inner Integral are function of y. That is,

$$\int_{c}^{d}\int_{k_1(y)}^{k_2(y)} f(x,y)\,dxdy$$

Step (i): First integrate $f(x,y)$ with respect to x (keeping y constant)

Step (ii) : Integrate the result obtained in step (i) with respect to y

Here order of integration is first w.r.t x and then w.r.t y (XY order)

For Example: Evaluate $\int_{0}^{1}\int_{0}^{\sqrt{1+y^2}} \frac{1}{1+x^2+y^2}dxdy$

Solution: Note that upper limit of inner integral is function of y. Therefore, order of integration is first w.r.t x and then w.r.t. y.

$$I=\int_{y=0}^{y=1}\int_{x=0}^{x=\sqrt{1+y^2}} \frac{1}{1+x^2+y^2}dxdy$$

Place the inner integral in parentheses so you can better see what you're working with:

$$I=\int_{y=0}^{y=1}\left(\int_{x=0}^{x=\sqrt{1+y^2}} \frac{1}{1+x^2+y^2}dx\right)dy$$

Now focus on what's inside the parentheses. For the moment, you can ignore the rest. Your integration variable is x, so treat the variable y as a constant:

$$I=\int_{y=0}^{y=1}\left(\int_{x=0}^{x=\sqrt{1+y^2}} \frac{1}{(1+y^2)+x^2}dx\right)dy$$

$$=\int_{y=0}^{y=1}\left[\frac{1}{\sqrt{1+y^2}}\tan^{-1}\left(\frac{x}{\sqrt{1+y^2}}\right)\right]_0^{\sqrt{1+y^2}}dy \qquad \left\{\text{Using } \int\frac{1}{a^2+x^2}=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right\}$$

$$=\int_{y=0}^{y=1}\frac{1}{\sqrt{1+y^2}}\left[\tan^{-1}\left(\frac{\sqrt{1+y^2}}{\sqrt{1+y^2}}\right)-\tan^{-1}(0)\right]dy$$

$$=\int_{y=0}^{y=1}\frac{1}{\sqrt{1+y^2}}\left[\tan^{-1}(1)-0\right]dy=\frac{\pi}{4}\int_{y=0}^{y=1}\frac{1}{\sqrt{1+y^2}}dy$$

$$=\frac{\pi}{4}\left[\log\left(y+\sqrt{1+y^2}\right)\right]_0^1=\frac{\pi}{4}\left[\log\left(1+\sqrt{1+1^2}\right)-\log(1)\right]=\frac{\pi}{4}\log\left(1+\sqrt{2}\right)$$

Type 1: Double Integrals with limits of integration

(1.1) Evaluate the following double integrals (Solved Examples)

1. $\int_{y=0}^{y=1} \int_{x=1}^{x=2} (x^2 y + xy)\,dxdy$

2. $\int_{x=0}^{x=\pi/4} \int_{y=0}^{y=\pi/2} \sin(x+y)\,dxdy$

3. $\int_{0}^{1} \int_{0}^{\sqrt{\frac{(1-y^2)}{2}}} \frac{1}{\sqrt{1-x^2-y^2}}\,dxdy$

4. $\int_{0}^{a\sqrt{3}} \int_{0}^{\sqrt{x^2+a^2}} \frac{x}{x^2+y^2+a^2}\,dxdy$

5. $\int_{0}^{a} \int_{0}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dxdy$

6. $\int_{0}^{2} \int_{0}^{x^2} x(x^2+y^2)\,dxdy$

7. $\int_{0}^{2} \int_{0}^{\sqrt{2y-y^2}} xy\,dxdy$

8. $\int_{0}^{1} \int_{0}^{x^2} e^{y/x}\,dxdy$

9. $\int_{0}^{1} \int_{0}^{y} xye^{-x^2}\,dxdy$

10. $\int_{0}^{1} \int_{0}^{1-y^2} \left[(x-1)^2+y^2\right]dxdy$

Practice Examples

11. $\int_{1}^{2} \int_{0}^{x} \frac{1}{x^2+y^2}\,dxdy$ Answer: $\frac{\pi}{4}\log 2$

12. $\int_{0}^{1} \int_{x}^{\sqrt{x}} \left[x^2+y^2\right]dxdy$ Answer: $\frac{3}{35}$

13. $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\,dxdy$ Answer: $\frac{\pi a^3}{6}$

14. $\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} y\ dxdy$ Answer: $\frac{ab^2}{3}$

15. $\int_0^1 \int_{x^2}^{\sqrt{x}} x^2 y\ dxdy$ Answer: $\frac{3}{56}$

16. $\int_1^4 \int_0^{\sqrt{y}} e^{\frac{x}{\sqrt{y}}} dxdy$ Answer: $\frac{14}{3}(e-1)$

Example 1.1.1: Evaluate $\int_{y=0}^{y=1}\int_{x=1}^{x=2}(x^2y+xy)\,dxdy$

Solution: Here order of integration is XY. That is, first integrate w.r.t x and then w.r.t. y.

$$I=\int_{y=0}^{y=1}\int_{x=1}^{x=2}y(x^2+x)\,dxdy$$

$$=\int_{y=0}^{y=1}y\left[\int_{x=1}^{x=2}(x^2+x)\,dx\right]dy$$

$$=\int_{y=0}^{y=1}y\left[\frac{x^3}{3}+\frac{x^2}{2}\right]_{x=1}^{x=2}dy$$

$$=\int_{y=0}^{y=1}y\left[\left(\frac{8}{3}+\frac{4}{2}\right)-\left(\frac{1}{3}+\frac{1}{2}\right)\right]dy$$

$$=\int_{y=0}^{y=1}y\left[\frac{23}{6}\right]dy$$

$$=\frac{23}{6}\left[\frac{y^2}{2}\right]_{y=0}^{y=1}$$

$$=\frac{23}{6}\left[\frac{1}{2}-0\right]=\frac{23}{12}$$

$$\therefore \boxed{I=\int_{y=0}^{y=1}\int_{x=1}^{x=2}y(x^2+x)\,dxdy=\frac{23}{12}}$$

Example 1.1.2: Evaluate $\int_{x=0}^{x=\pi/4}\int_{y=0}^{y=\pi/2}\sin(x+y)\,dxdy$

Solution: Here order of integration is YX. That is, first integrate w.r.t y and then w.r.t. x.

$$I=\int_{x=0}^{x=\pi/4}\int_{y=0}^{y=\pi/2}\sin(x+y)\,dxdy$$

$$=\int_{x=0}^{x=\pi/4}\left[\int_{y=0}^{y=\pi/2}\sin(x+y)\,dy\right]dx$$

$$=\int_{x=0}^{x=\pi/4}\left[-\cos(x+y)\right]_{y=0}^{y=\pi/2}dx$$

$$=-\int_{x=0}^{x=\pi/4}\left[\cos\left(x+\frac{\pi}{2}\right)-\cos x\right]dx$$

$$=-\int_{x=0}^{x=\pi/4}\left[-\sin x-\cos x\right]dx \qquad \left\{\because \cos\left(x+\frac{\pi}{2}\right)=-\sin x\right\}$$

$$=\int_{x=0}^{x=\pi/4}\left[\sin x+\cos x\right]dx$$

$$=\left[-\cos x+\sin x\right]_{x=0}^{x=\pi/4}$$

$$=\left\{\left[-\cos\left(\frac{\pi}{4}\right)+\sin\left(\frac{\pi}{4}\right)\right]-\left[-\cos 0+\sin 0\right]\right\}$$

$$=\left\{-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}+1-0\right\}=1$$

$$\therefore \boxed{\int_{x=0}^{x=\pi/4}\int_{y=0}^{y=\pi/2}\sin(x+y)\,dxdy=1}$$

Example 1.1.3: Evaluate $\int_0^1 \int_0^{\sqrt{\frac{(1-y^2)}{2}}} \frac{1}{\sqrt{1-x^2-y^2}}\,dxdy$

Solution: Upper limits of inner integral is function of y. Therefore, order of integration is first w.r.t x and then w.r.t. y. (XY order)

$$I = \int_{y=0}^{y=1} \int_{x=0}^{x=\sqrt{\frac{(1-y^2)}{2}}} \frac{1}{\sqrt{1-y^2-x^2}}\,dxdy$$

$$= \int_{y=0}^{y=1} \left[\int_{x=0}^{x=\sqrt{\frac{(1-y^2)}{2}}} \frac{1}{\sqrt{(1-y^2)-x^2}}\,dx \right] dy \quad \{\text{Integrating w.r.t. } x \text{ treating } y \text{ constant}\}$$

$$= \int_{y=0}^{y=1} \left[\sin^{-1}\left(\frac{x}{\sqrt{1-y^2}} \right) \right]_{x=0}^{x=\sqrt{\frac{(1-y^2)}{2}}} dy \qquad \left\{ \because \int \frac{1}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) \right\}$$

$$= \int_{y=0}^{y=1} \left[\sin^{-1}\left(\frac{1}{\sqrt{2}} \right) - \sin^{-1}(0) \right] dy$$

$$= \int_{y=0}^{y=1} \left[\frac{\pi}{4} - 0 \right] dy$$

$$= \frac{\pi}{4} [y]_{y=0}^{y=1}$$

$$= \frac{\pi}{4} [1-0] = \frac{\pi}{4}$$

$$\therefore \boxed{\int_{y=0}^{y=1} \int_{x=0}^{x=\sqrt{\frac{(1-y^2)}{2}}} \frac{1}{\sqrt{1-y^2-x^2}}\,dxdy = \frac{\pi}{4}}$$

Example 1.1.4: Evaluate $\int_0^{a\sqrt{3}}\int_0^{\sqrt{x^2+a^2}} \frac{x}{x^2+y^2+a^2}dxdy$

Solution: Upper limit of inner integral is function of x. Therefore, order of integration is first w.r.t y and then w.r.t x. (YX order)

$$I = \int_{x=0}^{x=a\sqrt{3}}\int_{y=0}^{y=\sqrt{x^2+a^2}} \frac{x}{x^2+y^2+a^2}dxdy$$

$$= \int_{x=0}^{x=a\sqrt{3}} x\left[\int_{y=0}^{y=\sqrt{x^2+a^2}} \frac{1}{\left(x^2+a^2\right)+y^2}dy\right]dx$$

$$= \int_{x=0}^{x=a\sqrt{3}} x\left[\frac{1}{\sqrt{x^2+a^2}}\tan^{-1}\left(\frac{y}{\sqrt{x^2+a^2}}\right)\right]_0^{\sqrt{x^2+a^2}} dx \qquad \left\{\because \int\frac{1}{b^2+y^2}dy=\frac{1}{b}\tan^{-1}\left(\frac{y}{b}\right)\right\}$$

$$= \int_{x=0}^{x=a\sqrt{3}} \frac{x}{\sqrt{x^2+a^2}}\left[\tan^{-1}(1)-\tan^{-1}(0)\right]dx$$

$$= \int_{x=0}^{x=a\sqrt{3}} \frac{x}{\sqrt{x^2+a^2}}\left[\frac{\pi}{4}\right]dx$$

Put $x^2+a^2=t \quad \therefore 2xdx=dt$

x	0	$a\sqrt{3}$
t	a^2	$4a^2$

$$\therefore I = \frac{\pi}{4}\int_{x=a^2}^{x=4a^2} \frac{1}{\sqrt{t}}\frac{dt}{2}$$

$$= \frac{\pi}{8}\left[2\sqrt{t}\right]_{a^2}^{4a^2}$$

$$= \frac{\pi}{4}[2a-a] = \frac{\pi a}{4}$$

$$\therefore \boxed{\int_{x=0}^{x=a\sqrt{3}}\int_{y=0}^{y=\sqrt{x^2+a^2}} \frac{x}{x^2+y^2+a^2}dxdy = \frac{\pi a}{4}}$$

Example 1.1.5: Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dxdy$

Solution: Upper limit of inner integral is function of x. Therefore, order of integration is first w.r.t y and then w.r.t. x. (YX order)

$$I = \int_{x=0}^{x=a} \int_{y=0}^{y=\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dxdy$$

$$= \int_{x=0}^{x=a} \left[\int_{y=0}^{y=\sqrt{a^2-x^2}} \sqrt{\left(a^2-x^2\right)-y^2}\,dy \right] dx$$

$$= \int_{x=0}^{x=a} \left[\frac{y}{2}\sqrt{\left(a^2-x^2\right)-y^2} + \frac{\left(a^2-x^2\right)}{2}\sin^{-1}\left(\frac{y}{\sqrt{a^2-x^2}}\right) \right]_{y=0}^{y=\sqrt{a^2-x^2}} dx$$

$$= \int_{x=0}^{x=a} \left[0 + \frac{\left(a^2-x^2\right)}{2} \cdot \sin^{-1}(1) - 0 - 0 \right] dx$$

$$= \int_{x=0}^{x=a} \left[\frac{a^2-x^2}{2} \cdot \frac{\pi}{2} \right] dx$$

$$= \frac{\pi}{4}\left[a^2x - \frac{x^3}{3} \right]_0^a$$

$$= \frac{\pi}{4}\left[a^3 - \frac{a^3}{3} - 0 - 0 \right]$$

$$= \frac{\pi}{4}\left[\frac{2}{3}a^3 \right] = \frac{\pi a^3}{6}$$

$$\therefore \boxed{\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dxdy = \frac{\pi a^3}{6}}$$

Example 1.1.6: Evaluate $\int_0^2 \int_0^{x^2} x\left(x^2+y^2\right) dxdy$

Solution: Upper limit of inner integral is function of x. Therefore, order of integration is first w.r.t y and then w.r.t x. (YX order)

$$I = \int_{x=0}^{x=2} \int_{y=0}^{y=x^2} x\left(x^2+y^2\right) dxdy$$

$$= \int_{x=0}^{x=2} \left[\int_{y=0}^{y=x^2} \left(x^3+xy^2\right) dy \right] dx$$

$$= \int_{x=0}^{x=2} \left[x^3 y + x\frac{y^3}{3} \right]_{y=0}^{y=x^2} dx$$

$$= \int_{x=0}^{x=2} \left[x^3x^2 + x\frac{x^6}{3} - 0 - 0 \right] dx$$

$$= \int_{x=0}^{x=2} \left[x^5 + \frac{x^7}{3} \right] dx$$

$$= \left[\frac{x^6}{6} + \frac{x^8}{24} \right]_0^2$$

$$= \left[\frac{2^6}{6} + \frac{2^8}{24} - 0 - 0 \right]$$

$$= \frac{32}{3} + \frac{32}{3} = \frac{64}{3}$$

$$\therefore \boxed{\int_0^2 \int_0^{x^2} x\left(x^2+y^2\right) dxdy = \frac{64}{3}}$$

Example 1.1.7: Evaluate $\int_{0}^{2}\int_{0}^{\sqrt{2y-y^2}} xy\,dx\,dy$

Solution: Upper limits of inner integral is function of y. Therefore, order of integration is first w.r.t x and then w.r.t y. (XY order)

$$I = \int_{y=0}^{y=2}\int_{x=0}^{x=\sqrt{2y-y^2}} xy\,dx\,dy$$

$$= \int_{y=0}^{y=2} y\left[\int_{x=0}^{x=\sqrt{2y-y^2}} x\,dx\right]dy$$

$$= \int_{y=0}^{y=2} y\left[\frac{x^2}{2}\right]_{x=0}^{x=\sqrt{2y-y^2}} dy$$

$$= \int_{y=0}^{y=2} \frac{y}{2}\left[2y-y^2\right]dy$$

$$= \int_{y=0}^{y=2}\left[y^2-\frac{y^3}{2}\right]dy$$

$$= \left[\frac{y^3}{3}-\frac{y^4}{8}\right]_{y=0}^{y=2}$$

$$= \left[\frac{8}{3}-\frac{16}{8}-0+0\right] = \frac{2}{3}$$

$$\therefore \boxed{\int_{y=0}^{y=2}\int_{x=0}^{x=\sqrt{2y-y^2}} xy\,dx\,dy = \frac{2}{3}}$$

Example 1.1.8: $\int_0^1 \int_0^{x^2} e^{y/x}\,dxdy$

Solution: Upper limit of inner integral is function of x. Therefore, order of integration is first w.r.t y and then w.r.t x. (YX order)

$$I = \int_{x=0}^{x=1} \int_{y=0}^{y=x^2} e^{y/x}\,dxdy$$

$$= \int_{x=0}^{x=1} \left[\int_{y=0}^{y=x^2} e^{y/x}\,dy \right] dx \text{--------}(1)$$

$$\text{Consider } I_1 = \int_{y=0}^{y=x^2} e^{y/x}\,dy = \int_{y=0}^{y=x^2} e^{\frac{1}{x}y}\,dy$$

$$= \left[\frac{e^{\frac{1}{x}y}}{\left(\frac{1}{x}\right)} \right]_{y=0}^{y=x^2} \quad \left\{ \because \int e^{ay}dy = \frac{e^{ay}}{a}, \quad \text{where } a = \frac{1}{x} \right\}$$

$$= \left[xe^{\frac{1}{x}y} \right]_{y=0}^{y=x^2} = x\left[e^x - 1\right]$$

$$\therefore I_1 = \int_{y=0}^{y=x^2} e^{y/x}\,dx = xe^x - x \text{----}(2)$$

From (1) and (2)

$$I = \int_{x=0}^{x=1} \left[xe^x - x \right] dx$$

$$= \left\{ x\left[e^x\right] - (1)\cdot\left[e^x\right] - \frac{x^2}{2} \right\}_{x=0}^{x=1} \quad \left\{ \text{Integrating } xe^x \text{ by parts using } \int uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots \right\}$$

$$= \left\{ \left[e - e - \frac{1}{2} \right] - \left[0 - 1 - 0\right] \right\} = \frac{1}{2}$$

$$\therefore \boxed{\int_{x=0}^{x=1} \int_{y=0}^{y=x^2} e^{y/x}\,dxdy = \frac{1}{2}}$$

Example 1.1.9: $\int_0^1 \int_0^y xye^{-x^2}dxdy$

Solution: Upper limit of inner integral is function of y. Therefore, order of integration is first w.r.t x and then w.r.t y (XY order)

$$I = \int_{y=0}^{y=1} \int_{x=0}^{x=y} xye^{-x^2}dxdy$$

$$= \int_{y=0}^{y=1} y\left[\int_{x=0}^{x=y} xe^{-x^2}dx\right]dy --------(1)$$

Consider $I_1 = \int_0^y xe^{-x^2}dx = \int_0^y e^{-x^2}(xdx)$

$$= \int_0^{y^2} e^{-t}\left(\frac{dt}{2}\right) \quad \left\{\text{Putting } x^2 = t \ \therefore 2xdx = dt \text{ and } \begin{array}{|c|c|c|}\hline x & 0 & y \\ \hline t & 0 & y^2 \\ \hline\end{array}\right\}$$

$$= \frac{1}{2}\left[-e^{-t}\right]_0^{y^2} = -\frac{1}{2}\left[e^{-y^2} - 1\right]$$

$$\therefore I_1 = \int_0^y xe^{-x^2}dx = \frac{1}{2}\left[1 - e^{-y^2}\right]----(2)$$

From (1) and (2)

$$\therefore I = \int_0^1 y\left[\frac{1}{2}\left(1 - e^{-y^2}\right)\right]dy$$

$$= \frac{1}{2}\int_0^1 \left(1 - e^{-y^2}\right)ydy$$

$$= \frac{1}{2}\int_0^1 \left(1 - e^{-t}\right)\left(\frac{dt}{2}\right) \quad \left\{\text{Putting } y^2 = t \ \therefore 2ydy = dt \text{ and } \begin{array}{|c|c|c|}\hline y & 0 & 1 \\ \hline t & 0 & 1 \\ \hline\end{array}\right\}$$

$$= \frac{1}{4}\left[t + e^{-t}\right]_0^1$$

$$= \frac{1}{4}\left[1 + e^{-1} - (0 + 1)\right] = \frac{1}{4}e^{-1} = \frac{1}{4e}$$

$$\therefore \boxed{I = \int_0^1 \int_0^y xye^{-x^2}dxdy = \frac{1}{4e}}$$

Example 1.1.10: Evaluate $\int_0^1 \int_0^{1-y^2} \left[(x-1)^2+y^2\right]dxdy$

Solution: Upper limit of inner integral is function of y. Therefore, order of integration is first w.r.t x and then w.r.t. y (XY order)

$$I = \int_{y=0}^{y=1} \int_{x=0}^{x=1-y^2} \left[(x-1)^2+y^2\right]dxdy$$

$$= \int_{y=0}^{y=1} \left[\int_{x=0}^{x=1-y^2} \left[(x-1)^2+y^2\right]dx \right]dy$$

$$= \int_{y=0}^{y=1} \left[\frac{(x-1)^3}{3} + y^2 x \right]_{x=0}^{x=1-y^2} dy$$

$$= \int_{y=0}^{y=1} \left\{ \left[\frac{1}{3}\left(1-y^2-1\right)^3 + y^2(1-y^2) \right] - \left[\frac{1}{3}(0-1)^3 + 0 \right] \right\} dy$$

$$= \int_{y=0}^{y=1} \left\{ \left[-\frac{1}{3}y^6 + y^2 - y^4 \right] - \left[-\frac{1}{3} \right] \right\} dy$$

$$= \left[-\frac{1}{3}\cdot\frac{y^7}{7} + \frac{y^3}{3} - \frac{y^5}{5} + \frac{1}{3}y \right]_{y=0}^{y=1}$$

$$= \left[-\frac{1}{21} + \frac{1}{3} - \frac{1}{5} + \frac{1}{3} \right] - 0$$

$$= \frac{44}{105}$$

$$\therefore \boxed{\int_{y=0}^{y=1} \int_{x=0}^{x=1-y^2} \left[(x-1)^2+y^2\right]dxdy = \frac{44}{105}}$$