Chapter-1	Double Integrals
Topic-1.2-A	

Type 2: Evaluation of Double Integral over given region

Working Rule to Evaluate Double Integrals when limits of integral are not given

1. Draw all the curves, find points of intersection and locate region of integration.

2. Decide the order of integration i.e. XY or YX

3. Function $f(x,y)$ will decide the order of integration. If integration of $f(x,y)$ w.r.t x (keeping y constant) is easy order is XY, otherwise order is YX.

Case 1: If order of integration is YX, then consider a strip parallel to Y axis.

(a) This strip is an imaginary strip of variable length which moves from left to right end of the region so that the entire region of integration is swept.

(b) During its journey from to left to right end, lower and upper end of the strip should touch single curve. Otherwise partition the region of integration.

(c) If lower end of the strip touches the curve $y = h(x)$, upper end touches the curve $y = k(x)$, X co-ordinate of a point on extreme left of the region is a and X coordinate of a point on extreme right of the region is b then

$$I = \int_{x=a}^{x=b} \int_{y=h(x)}^{y=k(x)} f(x,y)dxdy$$

Case 2: If order of integration is XY, then consider a strip parallel to X axis.

(a) This strip is an imaginary strip of variable length which moves from lower to upper end of the region so that the entire region of integration is swept.

(b) During its journey from to lower to upper end, left and right end of the strip should touch single curve. Otherwise partition the region of integration.

(c) If left end of the strip touches the curve $x = k_1(y)$, right end touches the curve $x = k_2(y)$, Y coordinate of a point at lower end of the region is c and Y coordinate of a point at upper end of the region is d then

$$I = \int_{y=c}^{y=d} \int_{x=k_1(y)}^{x=k_2(y)} f(x, y)dxdy$$

(1.2A): Examples

1. Evaluate $\iint_R e^{y/x}\,dxdy$, where region of integration R is the triangle bounded by the straight lines $y=0,\ x=1$ and $x=y$

2. Evaluate $\iint_R \sin[\pi(ax+by)]dxdy$, where R is the region bounded by the straight lines. $x=0,\ y=0$ and $ax+by=1$

3. Evaluate $\iint_R e^{ax+by}dxdy$ where R is the region bounded by the straight lines. $x=0,\ y=0$ and $ax+by=1$

4. Show that $\iint_R \left(x^3+y^2\right)dxdy=\frac{352}{15}$, where R is the triangle formed by the straight lines $y=0,\ x=2$ and $y=2x$

5. Evaluate $\iint_R xydxdy$, where R is the positive quadrant of the circle $x^2+y^2=a^2$

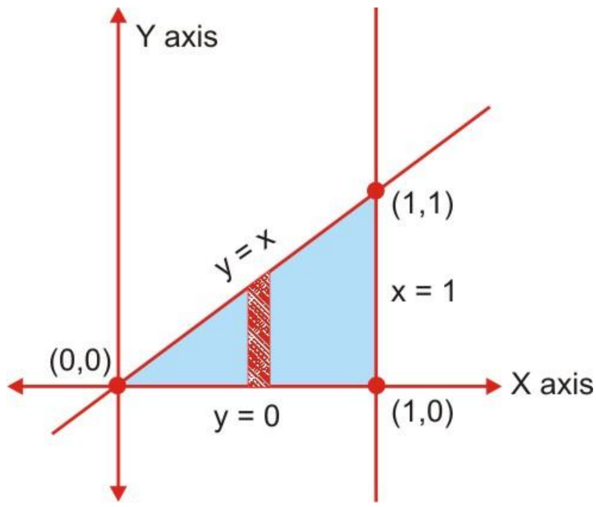
6. Evaluate $\iint_R xy(x+y)dxdy$, where R is the region bounded between $x^2=y$ & $x=y$

7. Evaluate $\iint_R \frac{1}{\sqrt{1-x^2-y^2}}dxdy$, where R is the region of ellipse $2x^2+y^2=1$ in the first quadrant.

For More Examples of similar type refer topic 1.2B

Example 1.2A.1: Evaluate $\iint_R e^{y/x}\,dxdy$, where region of integration R is the triangle bounded by the straight lines $y=0,\ x=1$ and $x=y$

Solution: $y=0$ is X axis, $x=1$ is a straight line parallel Y axis and passing through $(1,0)$ and $x=y$ is a straight line passing through $(0,0)$ and $(1,1)$



Integration is easy w.r.t y. So, let's evaluate using order YX. That is first integrate w.r.t y (keeping x constant) and then integrate w.r.t x.

To write limits, consider a strip **parallel to Y axis**, which moves from left to right end of the region of integration without changing the curves.

Lower end of the strip touches X axis ($y=0$). Therefore, lower limit of inner integral is $y=0$

Upper end of the strip touches the straight line $y=x$. Therefore, upper limit of inner integral is $y=x$.

Value of x at left end of the region is 0. Therefore, lower limit of outer integral is $x=0$

Value of x at right end of the region is 1 . Therefore, upper limit of outer integral is $x=1$.

$$I=\int_{x=0}^{x=1}\int_{y=0}^{y=x} e^{y/x}\,dydx = \int_{x=0}^{x=1}\left[\int_{y=0}^{y=x} e^{y/x}\,dy\right]dx$$

$$=\int_{x=0}^{x=1}\left[xe^{y/x}\right]_{y=0}^{y=x}dx \qquad \left\{\because \int e^{ay}dy=\frac{e^{ay}}{a},\ \text{where } a=\frac{1}{x}\right\}$$

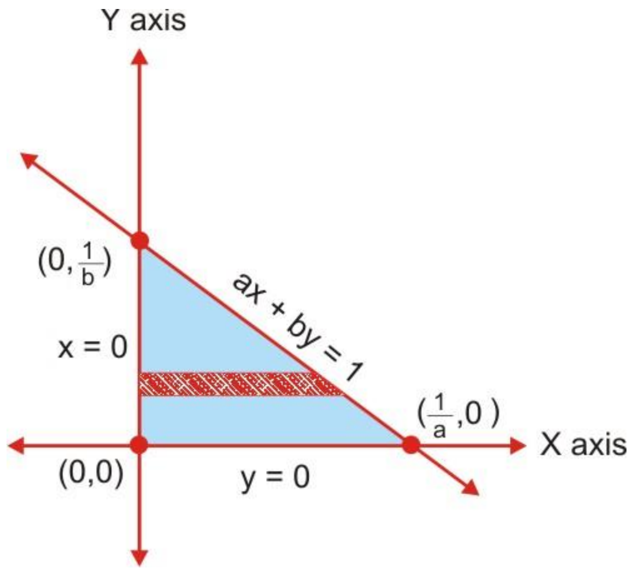
$$=\int_{x=0}^{x=1}\left[xe-x\right]dx$$

$$=\left[e\frac{x^2}{2}-\frac{x^2}{2}\right]_{x=0}^{x=1} = \frac{e}{2}-\frac{1}{2} = \frac{e-1}{2}$$

$$\therefore \boxed{\iint_R e^{y/x}\,dxdy=\frac{e-1}{2}}$$

Example 1.2A.2: Evaluate $\iint_R \sin[\pi(ax+by)]dxdy$, where R is the region bounded by the straight lines. $x=0,\ y=0$ and $ax+by=1$ (where a and b are positive constants).

Solution: $x=0$ is Y axis, $y=0$ is X axis and $ax+by=1$ is a straight line passing through $\left(\frac{1}{a},0\right)$ and $\left(0,\frac{1}{b}\right)$



Integration is easy w.r.t x as well as y. So, let's evaluate using order XY. That is first integrate w.r.t x (keeping y constant) and then integrate w.r.t y.

To write limits, consider a strip **parallel to X axis**, which moves from lower to upper end of the region of integration without changing the curves.

Left end of the strip touches Y axis ($x=0$). Therefore, lower limit of inner integral is $x=0$

Right end of the strip touches the straight line $ax+by=1$. Therefore, upper limit of inner integral is $y=\frac{1-ax}{b}$ (writing y in terms of x).

Value of y at lower end of the region is 0. Therefore, lower limit of outer integral is $y=0$

Value of y at upper end of the region is $\frac{1}{b}$. Therefore, upper limit of outer integral is $y=\frac{1}{b}$.

$$I = \int_{y=0}^{y=\frac{1}{b}} \int_{x=0}^{x=\left(\frac{1-by}{a}\right)} \sin\left[\pi ax+\pi by\right]dxdy$$

$$= \int_{y=0}^{y=\frac{1}{b}} \left[-\frac{\cos\left[\pi ax+\pi by\right]}{\pi a}\right]_{x=0}^{x=\left(\frac{1-by}{a}\right)} dy \quad \left\{\text{Integrating w.r.t. } x \ (\text{treating } y \text{ constant})\right\}$$

$$= -\frac{1}{\pi a}\int_{y=0}^{y=\frac{1}{b}} \left[\cos\left[\pi a\left(\frac{1-by}{a}\right)+\pi by\right]-\cos\left[\pi by\right]\right]dy$$

$$\therefore I = -\frac{1}{\pi a}\int_{y=0}^{y=\frac{1}{b}} \left[\cos[\pi] - \cos\left(\pi by\right)\right]dy$$

$$= -\frac{1}{\pi a}\int_{y=0}^{y=\frac{1}{b}} \left[-1 - \cos\left(\pi by\right)\right]dy$$

$$= \frac{1}{\pi a}\int_{y=0}^{y=\frac{1}{b}} \left[1 + \cos\left(\pi by\right)\right]dy$$

$$= \frac{1}{\pi a}\left[y + \frac{\sin\left(\pi by\right)}{\pi b}\right]_{y=0}^{y=\frac{1}{b}}$$

$$= \frac{1}{\pi a}\left[\frac{1}{b} + \frac{\sin\left(\pi\right)}{\pi b} - 0 - 0\right] = \frac{1}{\pi ab}$$

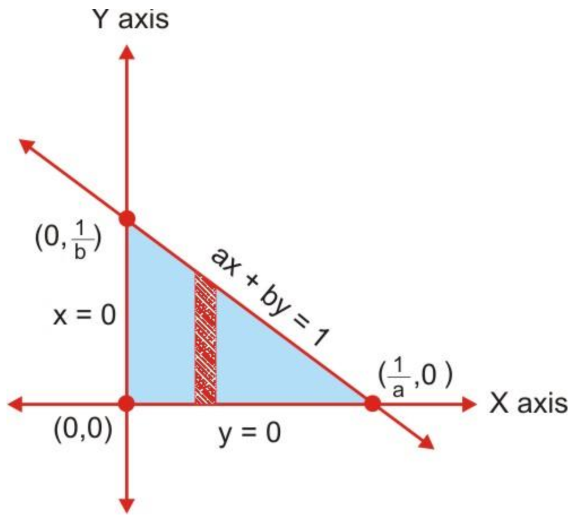
$$\therefore \boxed{\iint_R \sin[\pi(ax+by)]dxdy = \frac{1}{\pi ab}}$$

We can also evaluate this integral using order YX. That is, That is first integrating w.r.t y (keeping x constant) and then integrate w.r.t x. In this case, limits will be

$$I = \int_{x=0}^{x=\frac{1}{a}} \int_{y=0}^{y=\left(\frac{1-ax}{b}\right)} \sin\left[\pi ax + \pi by\right]dxdy$$

Example 1.2A.3: Evaluate $\iint_R e^{ax+by}dxdy$ where R is the region bounded by the straight lines. $x=0,\ y=0$ and $ax+by=1$ (where a and b are positive constants)

Solution: $x=0$ is Y axis, $y=0$ is X axis and $ax+by=1$ is a straight line passing through $\left(\frac{1}{a},0\right)$ and $\left(0,\frac{1}{b}\right)$



Integration is easy w.r.t x as well as y. So, let's evaluate using order YX. That is first integrate w.r.t y (keeping x constant) and then integrate w.r.t x.

To write limits, consider a strip **parallel to Y axis**, which moves from left to right end of the region of integration without changing the curves.

Lower end of the strip touches X axis ($y=0$). Therefore, lower limit of inner integral is $y=0$

Upper end of the strip touches the straight line $ax+by=1$. Therefore, upper limit of inner integral is $y=\frac{1-ax}{b}$ (writing y in terms of x).

Value of x at left end of the region is 0. Therefore, lower limit of outer integral is $x=0$

Value of x at right end of the region is $\frac{1}{a}$. Therefore, upper limit of outer integral is $x=\frac{1}{a}$.

$$I=\int_{x=0}^{x=\frac{1}{a}}\left[\int_{y=0}^{y=\left(\frac{1-ax}{b}\right)} e^{ax+by}dy\right]dx$$

$$=\int_{x=0}^{x=\frac{1}{a}}\left[\frac{e^{ax+by}}{b}\right]_{y=0}^{y=\left(\frac{1-ax}{b}\right)}dx \quad \left\{\text{Integrating w.r.t. } y \ \left(\text{treating } x \text{ constant}\right)\right\}$$

$$=\frac{1}{b}\int_{y=0}^{x=\frac{1}{a}}\left[e^{ax+b\left(\frac{1-ax}{b}\right)}-e^{ax}\right]dx$$

$$\therefore I = \frac{1}{b}\int_{y=0}^{x=\frac{1}{a}} \left[e - e^{ax}\right] dx$$

$$= \frac{1}{b}\int_{x=0}^{x=\frac{1}{a}} \left[e - e^{ax}\right] dx$$

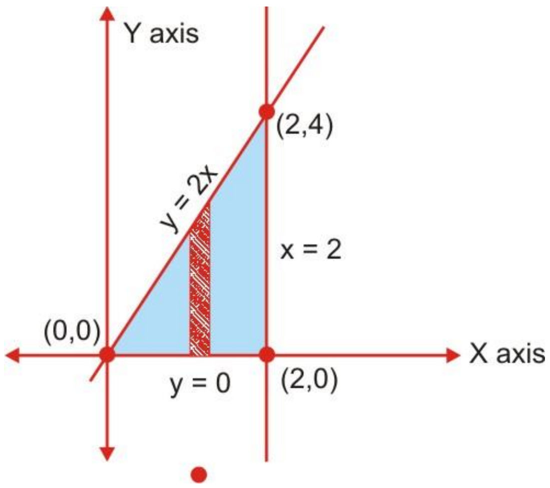
$$= \frac{1}{b}\left[ex - \frac{e^{ax}}{a}\right]_{x=0}^{x=\frac{1}{a}}$$

$$= \frac{1}{b}\left[\left(\frac{e}{a} - \frac{e}{a}\right) - \left(0 - \frac{1}{a}\right)\right] = \frac{1}{ab}$$

$$\therefore \boxed{\iint_R e^{ax+by}\,dxdy = \frac{1}{ab}}$$

Example 1.2A.4: Show that $\iint_R \left(x^3+y^2\right)dxdy=\frac{352}{15}$, where R is the triangle formed by the straight lines $y=0,\ x=2$ and $y=2x$

Solution: $y=0$ is X axis, $x=2$ is a straight line parallel to Y axis and passing through $(2,0)$ and $y=2x$ is a straight line passing through $(0,0)$ and $(2,4)$



Here we evaluate the integral using order YX. That is first integrate w.r.t y (keeping x constant) and then integrate w.r.t x. (We can also solve using order XY)

To write limits, consider a strip **parallel to Y axis**, which moves from left to right end of the region of integration without changing the curves.

Lower end of the strip touches X axis ($y=0$). Therefore, lower limit of inner integral is $y=0$

Upper end of the strip touches the straight line $y=2x$. Therefore, upper limit of inner integral is $y=2x$ (writing y in terms of x).

Value of x at left end of the region is 0. Therefore, lower limit of outer integral is $x=0$

Value of x at right end of the region is 2 . Therefore, upper limit of outer integral is $x=2$.

$$I=\int_{x=0}^{x=2}\left[\int_{y=0}^{y=2x}\left(x^3+y^2\right)dy\right]dx$$

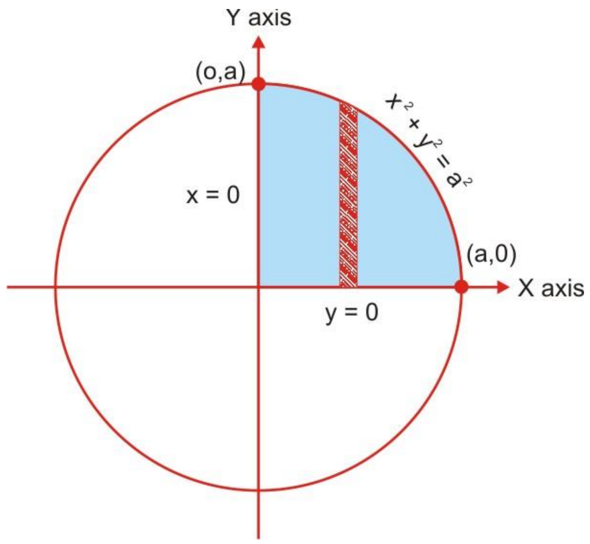
$$=\int_{x=0}^{x=2}\left[x^3y+\frac{y^3}{3}\right]_{y=0}^{y=2x}dx \quad \left\{\text{Integrating w.r.t. } y \ \left(\text{treating } x \text{ constant}\right)\right\}$$

$$=\int_{y=0}^{x=2}\left[2x^4+\frac{8x^3}{3}\right]dx=\left[2\frac{x^5}{5}+\frac{8}{3}\frac{x^4}{4}\right]_0^2=\frac{2}{5}(32)+\frac{2}{3}(16)=\frac{352}{15}$$

$$\therefore \boxed{\iint_R\left(x^3+y^2\right)dxdy=\frac{352}{15}}$$

Example 1.2A.5: Evaluate $\iint_R xydxdy$, where R is the positive quadrant of the circle $x^2 + y^2 = a^2$

Solution: The positive quadrant of the circle $x^2 + y^2 = a^2$ is bounded by the X axis, Y axis and arc of the circle $x^2 + y^2 = a^2$



Here we evaluate the integral using order YX. That is first integrate w.r.t y (keeping x constant) and then integrate w.r.t x. (We can also solve using order XY)

To write limits, consider a strip **parallel to Y axis**, which moves from left to right end of the region of integration without changing the curves.

$$I = \int_{x=0}^{x=a} \left[\int_{y=0}^{y=\sqrt{a^2-x^2}} xydy \right] dx$$

$$= \int_{x=0}^{x=a} \left[x\frac{y^2}{2} \right]_{y=0}^{y=\sqrt{a^2-x^2}} dx \quad \{ \text{Integrating w.r.t. } y \ (\text{treating } x \text{ constant}) \}$$

$$= \frac{1}{2}\int_{y=0}^{x=a} \left[x\left(\sqrt{a^2-x^2}\right)^2 \right] dx$$

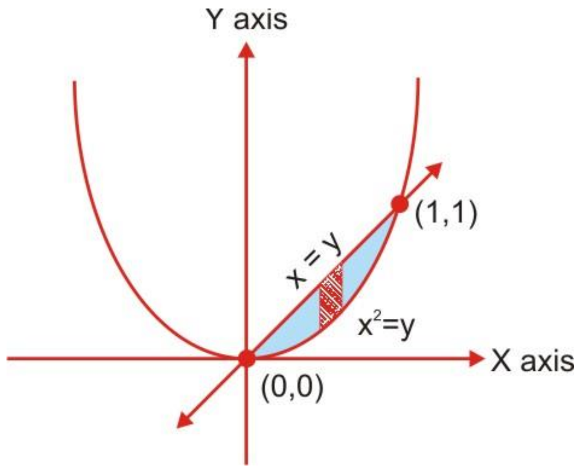
$$= \frac{1}{2}\int_{y=0}^{x=a} \left[a^2x - x^3 \right] dx$$

$$= \frac{1}{2}\left[a^2\frac{x^2}{2} - \frac{x^4}{4} \right]_{x=0}^{x=a} = \frac{1}{2}\left[\frac{a^4}{2} - \frac{a^4}{4} \right] = \frac{a^4}{8}$$

$$\therefore \boxed{\iint_R xydxdy = \frac{a^4}{8}}$$

Example 1.2A.6: Evaluate $\iint_R xy(x+y)dxdy$, where R is the region bounded between $x^2 = y$ & $x = y$

Solution: $x^2 = y$ is a parabola symmetrical bout Y axis, and $x = y$ is a straight line passing through $(0,0)$ and $(1,1)$



Let's evaluate the integral using order YX. That is first integrate w.r.t y (keeping x constant) and then integrate w.r.t x.

To write limits, consider a strip **parallel to Y axis**, which moves from left to right end of the region of integration without changing the curves.

Lower end of the strip touches the parabola $x^2 = y$. Therefore, lower limit of inner integral is $y = x^2$

Upper end of the strip touches the straight line $x = y$. Therefore, upper limit of inner integral is $x = y$ (writing x in terms of y).

Value of x at left end of the region is 0. Therefore, lower limit of outer integral is $x = 0$

Value of x at right end of the region is 1 . Therefore, upper limit of outer integral is $x = 1$.

$$I = \int_{x=0}^{x=1} \int_{y=x^2}^{y=x} xy\left(x+y\right)dxdy$$

$$= \int_{x=0}^{x=1} \left[\int_{y=x^2}^{y=x} \left(x^2y + xy^2\right)dx \right] dy$$

$$= \int_{x=0}^{x=1} \left[\frac{x^2y^2}{2} + \frac{xy^3}{3} \right]_{y=x^2}^{y=x} dx$$

$$= \int_{x=0}^{x=1} \left[\left(\frac{x^2x^2}{2} + \frac{xx^3}{3} \right) - \left(\frac{x^2x^4}{2} + \frac{xx^6}{3} \right) \right] dx$$

$$I = \int_{x=0}^{x=1} \left[\frac{5}{6}x^4 - \frac{1}{2}x^6 - \frac{1}{3}x^7 \right] dx$$

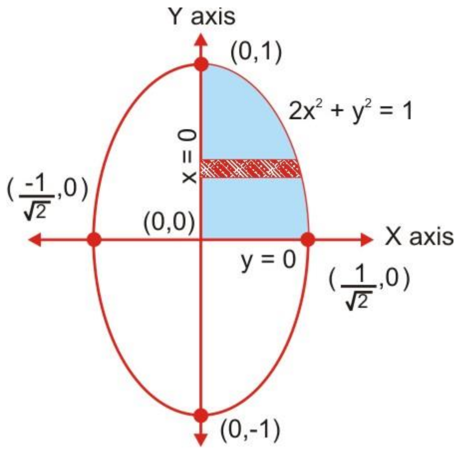
$$= \left[\frac{5}{6} \cdot \frac{x^5}{5} - \frac{1}{2} \cdot \frac{x^7}{7} - \frac{1}{3} \cdot \frac{x^8}{8} \right]_0^1$$

$$= \left[\frac{1}{6} - \frac{1}{14} - \frac{1}{24} \right] - 0 = \frac{3}{56}$$

$$\therefore \boxed{\iint xy\left(x+y\right) dxdy = \frac{3}{56}}$$

Example 1.2A.7: Evaluate $\iint_R \frac{1}{\sqrt{1-x^2-y^2}}dxdy$, where R is the region of ellipse $2x^2+y^2=1$ in the first quadrant.

Solution: The positive quadrant of the circle $2x^2+y^2=1$ is bounded by the X axis, Y axis and arc of the circle $2x^2+y^2=1$



Here we evaluate the integral using order XY. That is first integrate w.r.t x (keeping y constant) and then integrate w.r.t y.

To write limits, consider a strip **parallel to X axis**, which moves from lower to upper end of the region of integration without changing the curves.

$$2x^2+y^2=1 \Rightarrow x=\pm\sqrt{\frac{1-y^2}{2}}$$

$$I=\int_{y=0}^{y=1}\int_{x=0}^{x=\sqrt{\frac{(1-y^2)}{2}}}\frac{1}{\sqrt{1-y^2-x^2}}dxdy$$

$$=\int_{y=0}^{y=1}\left[\int_{x=0}^{x=\sqrt{\frac{(1-y^2)}{2}}}\frac{1}{\sqrt{(1-y^2)-x^2}}dx\right]dy$$

$$=\int_{y=0}^{y=1}\left[\sin^{-1}\left(\frac{x}{\sqrt{1-y^2}}\right)\right]_{x=0}^{x=\sqrt{\frac{(1-y^2)}{2}}}dy \quad \{\text{Integrating w.r.t. } x \text{ treating } y \text{ constant}\}$$

$$=\int_{y=0}^{y=1}\left[\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)-\sin^{-1}(0)\right]dy$$

$$=\int_{y=0}^{y=1}\left[\frac{\pi}{4}-0\right]dy \quad \left\{\because \cos\left(x+\frac{\pi}{2}\right)=-\sin x\right\}$$

$$=\frac{\pi}{4}[y]_{y=0}^{y=1}=\frac{\pi}{4}[1-0]=\frac{\pi}{4}$$

$$\therefore \boxed{\iint_R \frac{1}{\sqrt{1-x^2-y^2}}dxdy=\frac{\pi}{4}}$$