| Chapter-7 | Higher Order Linear Differential Equations with Constant Coefficients |
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| Topic-7.8 | |

# Cauchy's Differential Equations

Differential Equation of the form

$$x^3\frac{d^3y}{dx^3}+P_1\cdot x^2\frac{d^2y}{dx^2}+P_2\cdot x\frac{dy}{dx}+P_3\cdot y=Q(x),\ \text{where } P_1,P_2,P_3 \text{ are constants}$$

is called Cauchy's Differential Equation. Using substitution $x=e^z$ , we can transform above differential equation into Differential Equation with constant coefficients.

$$x=e^z \quad \therefore\ \log x=z \quad \therefore \frac{dz}{dx}=\frac{1}{x}$$

By chain rule $\frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx}=\frac{dy}{dz}\frac{1}{x}$,

$$\therefore x\frac{dy}{dx}=\frac{dy}{dz}=Dy,\ \text{where } D=\frac{d}{dz}$$

Similarly, $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$

$$=\frac{1}{x}\cdot\frac{d}{dx}\left(\frac{dy}{dz}\right)+\frac{dy}{dz}\cdot\frac{d}{dx}\left(\frac{1}{x}\right)$$

$$=\frac{1}{x}\cdot\frac{d}{dz}\left(\frac{dy}{dz}\right)\cdot\frac{dz}{dx}+\frac{dy}{dz}\cdot\left(-\frac{1}{x^2}\right)$$

$$=\frac{1}{x}\cdot\frac{d^2y}{dz^2}\cdot\left(\frac{1}{x}\right)-\frac{1}{x^2}\frac{dy}{dz}$$

$$\therefore \frac{d^2y}{dx^2}=\frac{1}{x^2}\frac{d^2y}{dz^2}-\frac{1}{x^2}\frac{dy}{dz}$$

Multiplying by $x^2$

$$x^2\frac{d^2y}{dx^2}=\frac{d^2y}{dz^2}-\frac{dy}{dz}=D^2y-Dy$$

$$\therefore x^2\frac{d^2y}{dx^2}=D\left(D-1\right)y$$

Similarly we can derive, $x^3\frac{d^3y}{dx^3}=D(D-1)(D-2)y$

Substituting $x\frac{dy}{dx}=Dy,\ x^2\frac{d^2y}{dx^2}=D(D-1)y,\ x^3\frac{d^3y}{dx^3}=D(D-1)(D-2)y$ in cauchy's differential equation, we get a linear differential equation with constant coefficients, which can be solved using methods discussed in previous sections.

## (7.8) Solve following differential equations

1. $x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+4y=\cos(\log x)+x\sin(\log x)$

2. $x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-20y=(x+1)^2$

3. $x^3\frac{d^3y}{dx^3}+3x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=x+\log x$

4. $x^2\frac{d^2y}{dx^2}+5x\frac{dy}{dx}+3y=\left(1+\frac{1}{x}\right)^2\cdot\log x$

5. $x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+y=\frac{\sin(\log x)+1}{x}$

6. $x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}-4y=x^2+2\log x$

7. $\left(\frac{d}{dx}+\frac{1}{x}\right)^2 y=\frac{1}{x^4}$

8. $x^2\frac{d^3y}{dx^3}+3x\frac{d^2y}{dx^2}+\frac{dy}{dx}=x^2\log x$

**Example 7.8.1:** Solve $x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+4y=\cos(\log x)+x\sin(\log x)$

**Solution**: Given Differential Equation is a Cauchy's Differential Equation

Put $x=e^z \quad \therefore z=\log x$ and $\frac{dz}{dx}=\frac{1}{x}$

By chain rule, $\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{dy}{dz}\cdot\frac{1}{x}$

$\therefore x\frac{dy}{dx}=\frac{dy}{dz}=Dy,$ where $D=\frac{d}{dz}$

Similarly, $x^2\frac{d^2y}{dx^2}=D(D-1)y$

Thus, given differential equation becomes

$D(D-1)y-Dy+4y=\cos z+e^z\sin z$

$\therefore (D^2-D-D+4)y=\cos z+e^z\sin z$

$\therefore (D^2-2D+4)y=\cos z+e^z\sin z$

Auxiliary Equation is $m^2-2m+4=0$

$\therefore m=\frac{2\pm\sqrt{4-16}}{2}=\frac{2\pm i2\sqrt{3}}{2}$

$\therefore m=1\pm i\sqrt{3}$ $\{$Roots are complex conjugate$\}$

Complementary Function $=e^z\left[c_1\cos\left(\sqrt{3}z\right)+c_2\sin\left(\sqrt{3}z\right)\right]$

$=x\left[c_1\cos\left(\sqrt{3}\log x\right)+c_2\sin\left(\sqrt{3}\log x\right)\right]$

$P.I=\frac{1}{D^2-2D+4}\{\cos z+e^z\sin z\}$

$$\therefore P.I = \frac{1}{D^2-2D+4}\cos z \; + \; \frac{1}{D^2-2D+4}e^z \sin z ---------(1)$$

$$= \frac{1}{-1^2-2D+4}\cos z \; + \; e^z\left[\frac{1}{(D+1)^2-2(D+1)+4}\sin z\right]$$

$$= \frac{1}{3-2D}\cos z \; + \; e^z\cdot\left[\frac{1}{D^2+2D+1-2D-2+4}\sin z\right]$$

$$= \frac{(3+2D)}{(3+2D)(3-2D)}\cos z \; + \; e^z\left[\frac{1}{D^2+3}\sin z\right]$$

$$= (3+2D)\left[\frac{1}{9-4D^2}\cos z\right] \; + \; e^z\left[\frac{1}{-1^2+3}\sin z\right]$$

$$= (3+2D)\left[\frac{1}{9-4(-1)}\cos z\right] + \frac{1}{2}e^z \sin z$$

$$= \frac{1}{13}(3+2D)\cos z + \frac{1}{2}e^z \sin z$$

$$= \frac{1}{13}(3\cos z + 2D\cos z) + \frac{1}{2}e^z \sin z$$

$$= \frac{1}{13}(3\cos z - 2\sin z) + \frac{1}{2}e^z \sin z$$

$$= \frac{1}{13}[3\cos(\log x) - 2\sin(\log x)] + \frac{1}{2}x\sin(\log x)$$

General Solution = Complementary Function + Particular Integral

$$\therefore \boxed{y = x\left[c_1\cos\left(\sqrt{3}\log x\right) + c_2\sin\left(\sqrt{3}\log x\right)\right] + \frac{1}{13}[3\cos(\log x) - 2\sin(\log x)] + \frac{1}{2}x\sin(\log x)}$$

**Example 7.8.2:** Solve $x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-20y=(x+1)^2$

**Solution**: Given Differential Equation is Cauchy's Differential Equation

Put $x=e^z \quad \therefore\ z=\log x$ and $\frac{dz}{dx}=\frac{1}{x}$

By chain rule, $\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{dy}{dz}\cdot\frac{1}{x}$

$\therefore x\frac{dy}{dx}=\frac{dy}{dz}=Dy,$ where $D=\frac{d}{dz}$

Similarly, $x^2\frac{d^2y}{dx^2}=D(D-1)y$

Thus, given differential equation becomes

$D(D-1)y+2Dy-20y=\left(e^z+1\right)^2$

$\therefore\left(D^2-D+2D-20\right)y=e^{2z}+2e^z+1$

$\therefore\left(D^2+D-20\right)y=e^{2z}+2e^z+1$

Auxiliary Equation is $m^2+m-20=0$

$\therefore(m+5)(m-4)=0$

$\therefore m=4,-5 \quad \{\text{Roots are real and distinct}\}$

Complementary Function $=c_1e^{4z}+c_2e^{-5z}=c_1x^4+c_2x^{-5}$

$P.I=\frac{1}{D^2+D-20}\left\{e^{2z}+2e^z+1\right\}$

$$\therefore P.I = \frac{1}{D^2+D-20}e^{2z} \; + \; 2\frac{1}{D^2+D-20}e^{z} \; + \; \frac{1}{D^2+D-20}e^{0z}$$

$$= \frac{1}{2^2+2-20}e^{2z} \; + \; 2\frac{1}{1^2+1-20}e^{z} \; + \; \frac{1}{0+-20}e^{0z}$$

$$= -\frac{1}{14}e^{2z} \; - \; \frac{1}{9}e^{z} \; - \; \frac{1}{20}$$

$$= -\frac{1}{14}x^2 \; - \; \frac{1}{9}x \; - \; \frac{1}{20}$$

General Solution = Complementary Function + Particular Integral

$$\therefore \boxed{y = c_1x^4 + c_2x^{-5} - \frac{1}{14}x^2 - \frac{1}{9}x - \frac{1}{20}}$$

**Example 7.8.3:** Solve $x^3\frac{d^3y}{dx^3}+3x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=x+\log x$

**Solution**: Given Differential Equation is Cauchy's Differential Equation

Put $x=e^z \quad \therefore z=\log x$ and $\frac{dz}{dx}=\frac{1}{x}$

By chain rule, $\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{dy}{dz}\cdot\frac{1}{x}$

$\therefore x\frac{dy}{dx}=\frac{dy}{dz}=Dy,$ where $D=\frac{d}{dz}$

Similarly, $x^2\frac{d^2y}{dx^2}=D(D-1)y$

and $x^3\frac{d^3y}{dx^3}=D(D-1)(D-2)y$

$$x^3\frac{d^3y}{dx^3}+3x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=x+\log x$$

Thus, given differential equation becomes

$$D(D-1)(D-2)y+3D(D-1)y+Dy+y=e^z+z$$

$$\therefore \left(D^3-3D^2+2D\right)y+\left(3D^2-3D\right)y+Dy+y=e^z+z$$

$$\therefore \left(D^3-3D^2+2D+3D^2-3D+D+1\right)y=e^z+z$$

$$\therefore \left(D^3+1\right)y=e^z+z$$

Auxiliary Equation is $m^3+1=0$

$$\therefore (m+1)\left(m^2-m+1\right)=0$$

$$\therefore m+1=0,\quad m^2-m+1=0$$

$$\therefore m=-1,\ m=\frac{1\pm\sqrt{1-4}}{2}=\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$$

$$\text{Complementary Function}=c_1e^{-z}+e^{z/2}\left(c_2\cos\left(\frac{\sqrt{3}}{2}z\right)+c_3\sin\left(\frac{\sqrt{3}}{2}z\right)\right)$$

$$=c_1x^{-1}+\sqrt{x}\left[c_2\cos\left(\frac{\sqrt{3}}{2}\log x\right)+c_3\sin\left(\frac{\sqrt{3}}{2}\log x\right)\right]$$

$$P.I = \frac{1}{D^3+1}\left\{e^z + z\right\}$$

$$= \frac{1}{D^3+1}e^z \ + \ \frac{1}{D^3+1}z$$

$$= \frac{1}{1^3+1}e^z \ + \ \frac{1}{1+D^3}z$$

$$= \frac{1}{2}e^z \ + \ \left[1 - D^3 + \cdots\right]z$$

$$= \frac{1}{2}e^z \ + \ z$$

$$= \frac{1}{2}x \ + \ \log x$$

General Solution = Complementary Function + Particular Integral

$$\therefore \boxed{y = c_1 x^{-1} + \sqrt{x}\left[c_2\cos\left(\frac{\sqrt{3}}{2}\log x\right) + c_3\sin\left(\frac{\sqrt{3}}{2}\log x\right)\right] + \frac{1}{2}x + \log x}$$

**Example 7.8.4:** Solve $x^2\frac{d^2y}{dx^2}+5x\frac{dy}{dx}+3y=\left(1+\frac{1}{x}\right)^2\cdot\log x$

**Solution**: Given Differential Equation is Cauchy's Differential Equation

Put $x=e^z \quad \therefore z=\log x$ and $\frac{dz}{dx}=\frac{1}{x}$

By chain rule, $\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{dy}{dz}\cdot\frac{1}{x}$

$\therefore x\frac{dy}{dx}=\frac{dy}{dz}=Dy,$ where $D=\frac{d}{dz}$

Similarly, $x^2\frac{d^2y}{dx^2}=D(D-1)y$

$$x^2\frac{d^2y}{dx^2}+5x\frac{dy}{dx}+3y=\left(1+\frac{1}{x}\right)^2\cdot\log x$$

Thus, given differential equation becomes

$$D(D-1)y+5Dy+3y=\left(1+\frac{1}{e^z}\right)^2\cdot z$$

$$\therefore\left(D^2-D+5D+3\right)y=\left(1+2e^{-z}+e^{-2z}\right)z$$

$$\therefore\left(D^2+4D+3\right)y=z+2e^{-z}z+e^{-2z}z$$

Auxiliary Equation is $m^2+4m+3=0$

$\therefore(m+3)(m+1)=0$

$\therefore m=-3,-1 \quad$ {Roots are real and distinct}

Complementary Function $=c_1e^{-z}+c_2e^{-3z}=c_1x^{-1}+c_2x^{-3}$

$$P.I=\frac{1}{D^2+4D+3}\left\{z+2e^{-z}z+e^{-2z}z\right\}$$

$$=\frac{1}{D^2+4D+3}z\;+\;2\frac{1}{D^2+4D+3}e^{-z}z\;+\;\frac{1}{D^2+4D+3}e^{-2z}z------(1)$$

Consider $I_1 = \frac{1}{D^2+4D+3}z = \frac{1}{3\left(1+\frac{D^2+4D}{3}\right)}z = \frac{1}{3}\left[1-\frac{D^2+4D}{3}+\cdots\right]z$

$$= \frac{1}{3}\left[1-\frac{4}{3}D+\cdots\right]z = \frac{1}{3}\left[z-\frac{4}{3}\right]$$

$$\therefore \frac{1}{D^2+4D+3}z = \left(\frac{z}{3}-\frac{4}{9}\right)--------(2)$$

$$I_2 = \frac{1}{D^2+4D+3}e^{-z}z = e^{-z}\frac{1}{(D-1)^2+4(D-1)+3}z = e^{-z}\frac{1}{D^2-2D+1+4D-4+3}z$$

$$= e^{-z}\frac{1}{D^2+2D}z = e^{-z}\frac{1}{2D\left(1+\frac{D}{2}\right)}z$$

$$= \frac{e^{-z}}{2}\frac{1}{D}\left[1-\frac{D}{2}+\cdots\right]z = \frac{e^{-z}}{2}\frac{1}{D}\left[z-\frac{1}{2}\right] = \frac{e^{-z}}{2}\left[\frac{z^2}{2}-\frac{z}{2}\right]$$

$$\therefore \frac{1}{D^2+4D+3}e^{-z}z = \frac{e^{-z}}{4}\left[z^2-z\right]----------(3)$$

$$I_3 = \frac{1}{D^2+4D+3}e^{-2z}z = e^{-2z}\frac{1}{(D-2)^2+4(D-2)+3}z = e^{-2z}\frac{1}{D^2-4D+4+4D-8+3}z$$

$$= e^{-2z}\frac{1}{D^2-1}z = e^{-2z}\frac{1}{-\left(1-D^2\right)}z = -e^{-2z}\left[1+D^2+\cdots\right]z = -e^{-2z}z$$

$$\frac{1}{D^2+4D+3}e^{-2z}z = -e^{-2z}z----------------(4)$$

Substituting from (2), (3) and (4) in (1)

$$P.I = \left(\frac{z}{3}-\frac{4}{9}\right)+2\cdot\frac{e^{-z}}{4}\left[z^2-z\right]-e^{-2z}z = \frac{\log x}{3}-\frac{4}{9}-\frac{1}{2x}\left((\log x)^2-\log x\right)-\frac{\log x}{x^2}$$

General Solution = Complementary Function + Particular Integral

$$\therefore \boxed{y = \left[\frac{c_1}{x}+\frac{c_2}{x^3}\right]+\left[\frac{\log x}{3}-\frac{4}{9}-\frac{1}{2x}\left((\log x)^2-\log x\right)-\frac{\log x}{x^2}\right]}$$

**Example 7.8.5:** Solve $x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+y=\frac{\sin(\log x)+1}{x}$

**Solution**: Given Differential Equation is Cauchy's Differential Equation

Put $x=e^z \quad \therefore z=\log x$ and $\frac{dz}{dx}=\frac{1}{x}$

By chain rule, $\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{dy}{dz}\cdot\frac{1}{x}$

$\therefore x\frac{dy}{dx}=\frac{dy}{dz}=Dy,$ where $D=\frac{d}{dz}$

Similarly, $x^2\frac{d^2y}{dx^2}=D(D-1)y$

Thus, given differential equation becomes

$$D(D-1)y-3Dy+y=\frac{\sin z+1}{e^z}$$

$$\therefore \left(D^2-D-3D+1\right)y=e^{-z}\sin z+e^{-z}$$

$$\therefore \left(D^2-4D+1\right)y=e^{-z}\sin z+e^{-z}$$

Auxiliary Equation is $m^2-4m+1=0$

$$\therefore m=\frac{4\pm\sqrt{16-4}}{2}=\frac{4\pm2\sqrt{3}}{2}=$$

$\therefore m=2\pm\sqrt{3}$ $\quad$ {Roots are real and distinct}

Complementary Function $=c_1e^{\left(2+\sqrt{3}\right)z}+c_2e^{\left(2-\sqrt{3}\right)z}=c_1x^{\left(2+\sqrt{3}\right)}+c_2x^{\left(2-\sqrt{3}\right)}$

$$P.I=\frac{1}{D^2-4D+1}\left\{e^{-z}\sin z+e^{-z}\right\}$$

$$=\frac{1}{D^2-4D+1}e^{-z}\sin z\ +\ \frac{1}{D^2-4D+1}e^{-z}\ -----(1)$$

Consider $I_1 = \dfrac{1}{D^2-4D+1}e^{-z}\sin z = e^{-z}\dfrac{1}{(D-1)^2-4(D-1)+1}\sin z$

$$= e^{-z}\frac{1}{D^2-2D+1-4D+4+1}\sin z$$

$$= e^{-z}\frac{1}{D^2-6D+6}\sin z$$

$$= e^{-z}\frac{1}{-1-6D+6}\sin z \quad \left\{\text{Type 2, Replacing each } D^2 \text{ by } -1^2=-1\right\}$$

$$= e^{-z}\frac{1}{5-6D}\sin z$$

$$= e^{-z}\frac{(5+6D)}{(5+6D)(5-6D)}\sin z$$

$$= e^{-z}\frac{(5+6D)}{25-36D^2}\sin z$$

$$= e^{-z}\frac{(5+6D)}{25-36(-1)}\sin z \quad \left\{\text{Type 2, Replacing each } D^2 \text{ by } -1^2=-1\right\}$$

$$= \frac{e^{-z}}{61}(5\sin z + 6D\sin z)$$

$$\therefore \frac{1}{D^2-4D+1}e^{-z}\sin z = \frac{e^{-z}}{61}(5\sin z + 6\cos z) --------(2)$$

$$I_2 = \frac{1}{D^2-4D+1}e^{-z} = \frac{1}{(-1)^2-4(-1)+1}e^{-z} = \frac{1}{6}e^{-z} -------(3)$$

Substituting from (2) and (3) in (1)

$$P.I = \frac{e^{-z}}{61}(5\sin z + 6\cos z) + \frac{1}{6}e^{-z} = \frac{e^{-z}}{61x}\left[5\sin(\log x) + 6\cos(\log x)\right] + \frac{1}{6x}$$

General Solution = Complementary Function + Particular Integral

$$\therefore \boxed{y = c_1 x^{(2+\sqrt{3})} + c_2 x^{(2-\sqrt{3})} + \frac{e^{-z}}{61x}\left[5\sin(\log x) + 6\cos(\log x)\right] + \frac{1}{6x}}$$

**Example 7.8.6:** Solve $x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}-4y=x^2+2\log x$

**Solution**: Given Differential Equation is Cauchy's Differential Equation

Put $x=e^z$ $\quad\therefore z=\log x$ and $\frac{dz}{dx}=\frac{1}{x}$

By chain rule, $\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{dy}{dz}\cdot\frac{1}{x}$

$\therefore x\frac{dy}{dx}=\frac{dy}{dz}=Dy,\quad$ where $D=\frac{d}{dz}$

Similarly, $x^2\frac{d^2y}{dx^2}=D(D-1)y$

$$x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}-4y=x^2+2\log x$$

Thus, given differential equation becomes

$$D(D-1)y-2Dy-4y=e^{2z}+2z$$

$$\therefore\left(D^2-D-2D-4\right)y=e^{2z}+2z$$

$$\therefore\left(D^2-3D-4\right)y=e^{2z}+2z$$

Auxiliary Equation is $m^2-3m-4=0$

$\therefore(m+1)(m-4)$

$\therefore m=-1,4\quad$ {Roots are real and distinct}

Complementary Function $=c_1e^{-z}+c_2e^{4z}=c_1x^{-1}+c_2x^4$

$$P.I=\frac{1}{D^2-3D-4}\left\{e^{2z}+2z\right\}$$

$$=\frac{1}{D^2-3D-4}e^{2z}\;+\;2\frac{1}{D^2-3D-4}z-----(1)$$

Consider $I_1=\frac{1}{D^2-3D-4}e^{2z}=\frac{1}{2^2-3(2)-4}e^{2z}=\frac{1}{4-6-4}e^{2z}=-\frac{1}{6}e^{2z}$

$$\therefore\frac{1}{D^2-3D-4}e^{2z}=-\frac{1}{6}e^{2z}-------------(2)$$

Consider $I_2 = \dfrac{1}{D^2-3D-4}z$

$$= \frac{1}{-4\left(1-\dfrac{D^2-3D}{4}\right)}z$$

$$= -\frac{1}{4}\left[1+\left(\frac{D^2-3D}{4}\right)+\cdots\right]z$$

$$= -\frac{1}{4}\left[1-\frac{3}{4}D+\cdots\right]z$$

$$= -\frac{1}{4}\left[z-\frac{3}{4}\right]$$

$$\therefore \frac{1}{D^2-3D-4}z = \left(-\frac{z}{4}+\frac{3}{16}\right)------(3)$$

Substituting from (2) and (3) in (1)

$$P.I = -\frac{1}{6}e^{2z}+2\left(-\frac{z}{4}+\frac{3}{16}\right) = -\frac{1}{6}e^{2z}-\frac{z}{2}+\frac{3}{8} = -\frac{1}{6}x^2-\frac{1}{2}\log x+\frac{3}{8}$$

General Solution = Complementary Function + Particular Integral

$$\therefore \boxed{y=\left[\frac{c_1}{x}+c_2x^4\right]+\left[-\frac{1}{6}x^2-\frac{1}{2}\log x+\frac{3}{8}\right]}$$

**Example 7.8.7:** Solve $\left(\frac{d}{dx}+\frac{1}{x}\right)^2 y=\frac{1}{x^4}$

**Solution**: Simplifying given differential equation

$$\therefore\left(\frac{d}{dx}+\frac{1}{x}\right)\left(\frac{d}{dx}+\frac{1}{x}\right)y=\frac{1}{x^4}$$

$$\therefore\left(\frac{d}{dx}+\frac{1}{x}\right)\left(\frac{dy}{dx}+\frac{y}{x}\right)=\frac{1}{x^4}$$

$$\therefore\frac{d}{dx}\left(\frac{dy}{dx}+\frac{y}{x}\right)+\frac{1}{x}\left(\frac{dy}{dx}+\frac{y}{x}\right)=\frac{1}{x^4}$$

$$\therefore\frac{d^2y}{dx^2}+\frac{d}{dx}\left(\frac{y}{x}\right)+\frac{1}{x}\frac{dy}{dx}+\frac{y}{x^2}=\frac{1}{x^4}$$

$$\therefore\frac{d^2y}{dx^2}+\left[\frac{x\frac{dy}{dx}-y}{x^2}\right]+\frac{1}{x}\frac{dy}{dx}+\frac{y}{x^2}=\frac{1}{x^4}$$

$$\therefore\frac{d^2y}{dx^2}+\left[\frac{1}{x}\frac{dy}{dx}-\frac{y}{x^2}\right]+\frac{1}{x}\frac{dy}{dx}+\frac{y}{x^2}=\frac{1}{x^4}$$

$$\therefore\frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}=\frac{1}{x^4}$$

Multiplying by $x^2$

$$\therefore x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}=\frac{1}{x^2}\text{--------}(1)$$

Put $x=e^z$ $\therefore$ $z=\log x$ and $\frac{dz}{dx}=\frac{1}{x}$

By chain rule, $\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{dy}{dz}\cdot\frac{1}{x}$

$\therefore x\frac{dy}{dx}=\frac{dy}{dz}=Dy,$ where $D=\frac{d}{dz}$

Similarly, $x^2\frac{d^2y}{dx^2}=D(D-1)y$

Thus, D.E (1) becomes $D(D-1)y+2Dy=\frac{1}{e^{2z}}$

$$\therefore\left(D^2-D+2D\right)y=e^{-2z}$$

$$\therefore\left(D^2+D\right)y=e^{-2z}$$

Auxiliary Equation is $m^2 + m = 0$

$\therefore m(m+1) = 0$

$\therefore m = 0, -1$ $\{$Roots are real and distinct$\}$

Complementary Function $= c_1 e^{0z} + c_2 e^{-z} = c_1 + c_2 x^{-1} = c_1 + \frac{c_2}{x}$

$$P.I = \frac{1}{D^2 + D} e^{-2z}$$

$$= \frac{1}{(-2)^2 + (-2)} e^{-2z}$$

$$= \frac{1}{2} e^{-2z} = \frac{1}{2x^2}$$

General Solution = Complementary Function + Particular Integral

$$\therefore \boxed{y = c_1 + \frac{c_2}{x} + \frac{1}{2x^2}}$$

**Example 7.8.8:** Solve $x^2\frac{d^3y}{dx^3}+3x\frac{d^2y}{dx^2}+\frac{dy}{dx}=x^2\log x$

**Solution**: Multiplying given differential equation by $x$, $x^3\frac{d^3y}{dx^3}+3x^3\frac{d^2y}{dx^2}+x\frac{dy}{dx}=x^3\log x$

$$x^3\frac{d^3y}{dx^3}+3x^3\frac{d^2y}{dx^2}+x\frac{dy}{dx}=x^3\log x$$

Put $x=e^z \quad \therefore z=\log x$ and $\frac{dz}{dx}=\frac{1}{x}$

By chain rule, $\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{dy}{dz}\cdot\frac{1}{x}$

$\therefore x\frac{dy}{dx}=\frac{dy}{dz}=Dy$, where $D=\frac{d}{dz}$

Similarly, $x^2\frac{d^2y}{dx^2}=D(D-1)y$

and $x^3\frac{d^3y}{dx^3}=D(D-1)(D-2)y$

Thus, given differential equation becomes

$$D(D-1)(D-2)y+\ 3D(D-1)y+Dy=e^{3z}z$$

$$\therefore\left(D^3-3D^2+2D+3D^2-3D+D\right)y=e^{3z}z$$

$$\therefore D^3y=e^{3z}z$$

Auxiliary Equation is $m^3=0$

$\therefore m=0,0,0 \quad$ {Roots are real and equal}

Complementary Function $=\left(c_1+c_2z+c_3z^2\right)e^{0z}=c_1+c_2\log x+c_3(\log x)^2$

$$P.I=\frac{1}{D^3}ze^{3z}=\left\{z-\frac{3D^2}{D^3}\right\}\frac{1}{D^3}e^{3z}$$

$$=\left\{z-\frac{3}{D}\right\}\frac{1}{3^3}e^{3z}$$

$$=\frac{1}{27}\left\{ze^{3z}-3\frac{1}{D}e^{3z}\right\}$$

$$\therefore P.I = \frac{1}{27}\left\{ze^{3z} - 3\frac{1}{D}e^{3z}\right\}$$

$$= \frac{1}{27}\left\{ze^{3z} - 3\frac{1}{3}e^{3z}\right\}$$

$$= \frac{1}{27}\left\{ze^{3z} - e^{3z}\right\}$$

$$= \frac{e^{3z}}{27}(z-1)$$

$$= \frac{x^3}{27}(\log x - 1)$$

General Solution=Comlementary Function + Particular Integral

$$\boxed{G.S = c_1 + c_2 \log x + c_3 (\log x)^2 + \frac{x^3}{27}(\log x - 1)}$$