Chapter-5	First Order and First Degree Ordinary Differential Equations
Topic-5.6	

Type 6: Bernoulli's Differential Equation

Differential equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)\cdot y^n \text{ -------------------------- } (1)$$

where $P(x)$ and $Q(x)$ are functions of x, is called **Bernoulli's Differential Equation**.

To solve Bernoulli's Differential Equation, dividing it by y^n

$$\frac{1}{y^n}\frac{dy}{dx} + P(x)\frac{1}{y^{n-1}} = Q(x)$$

Put $\frac{1}{y^{n-1}} = y^{1-n} = V$

$\therefore (1-n)\, y^{1-n-1}\frac{dy}{dx} = \frac{dV}{dx}$ or $\frac{1}{y^n}\frac{dy}{dx} = \frac{1}{(1-n)}\frac{dV}{dx}$

Thus, Bernoulli's Differential Equation (1) reduces to $\frac{1}{(1-n)}\frac{dV}{dx} + P(x)V = Q(x)$

Multiplying throughout by $(1-n)$, we get

$$\frac{dV}{dx} + (1-n)P(x)V = (1-n)Q(x)$$

This is a first order linear differential equation, which can be solved using the method discussed in the previous section.

Another type of differential equation, which can be reduced to first order linear differential equation is

$$f'(y)\frac{dy}{dx} + P(x)\cdot f(y) = Q(x) \text{ -------------------------- } (2)$$

Here, $f(y) = V$, $\therefore f'(y)\frac{dy}{dx} = \frac{dV}{dx}$

Thus, Differential Equation (2) reduces to $\frac{dV}{dx} + P(x)\cdot V = Q(x)$

This is a first order linear differential equation.

(5.6) Solve following differential equations

1. $\frac{dy}{dx}+\frac{y}{3}=e^{x}y^{4}$

2. $\frac{dy}{dx}+\frac{2y}{x}=-x^{2}y^{2}\cos x$

3. $\frac{dy}{dx}-xy=y^{2}e^{-x^{2}/2}\log x$

4. $2\frac{dy}{dx}+y\tan x=\frac{(4x+5)^{2}}{\cos x}y^{3}$

5. $\frac{dy}{dx}=x^{3}y^{3}-xy$

6. $xy(1+xy^{2})\frac{dy}{dx}=1$

7. $y\frac{dy}{dx}+\frac{4x}{3}-\frac{y^{2}}{3x}=0$

8. $\sec^{2}y\frac{dy}{dx}+2\tan x\cdot\tan y=\sin x$

9. $\frac{dy}{dx}=e^{x-y}(e^{x}-e^{y})$

10. $\frac{dy}{dx}+(2x\tan^{-1}y-x^{3})(1+y^{2})=0$

11. $\frac{dr}{d\theta}=\frac{r\sin\theta-r^{2}}{\cos\theta}$

12. $\frac{dy}{dx}+\frac{y}{x}\log y=\frac{y}{x^{2}}(\log y)^{2}$

13. $\frac{dy}{dx}=1-x(y-x)-x^{3}(y-x)^{2}$

14. $(1+\sin y)\frac{dx}{dy}=[2y\cos y-x(\sec y+\tan y)]$

15. $\frac{dy}{dx}+x^{3}\sin^{2}y+x\sin 2y=x^{3}$

Example 5.6.1: Solve $\frac{dy}{dx}+\frac{y}{3}=e^{x}y^{4}$

Solution: Dividing both the sides of differential equation by y^4

$$\frac{1}{y^4}\frac{dy}{dx}+\frac{1}{3}\frac{1}{y^3}=e^{x} ------------(1)$$

Put $\frac{1}{y^3}=V,\quad \therefore -\frac{3}{y^4}\frac{dy}{dx}=\frac{dV}{dx}\ or\ \frac{1}{y^4}\frac{dy}{dx}=-\frac{1}{3}\frac{dV}{dx}$

Therefore, from (1) $-\frac{1}{3}\frac{dV}{dx}+\frac{1}{3}V=e^{x}$

Multiplying throughout by -3

$$\frac{dV}{dx}-V=-3e^{x}$$

This is a first order linear differential equation of the form $\frac{dV}{dx}+P(x)V=Q(x)$

where, $P(x)=-1$ and $Q(x)=-3e^{x}$

$$\int P(x)\,dx=\int(-1)dx=-x$$

$$\therefore I.F=e^{\int P(x)dx}=e^{-x}$$

Its solution is $V\cdot(I.F)=\int(I.F)\cdot Q(x)\,dx+C$

$$\therefore V\cdot e^{-x}=\int e^{-x}(-3e^{x})dx+C$$

$$\therefore \frac{1}{y^3}\cdot e^{-x}=\int -3dx+C \quad \left\{\because V=\frac{1}{y^3}\right\}$$

$$\therefore \boxed{\frac{1}{y^3}\cdot e^{-x}=-3x+C}$$

Example 5.6.2: Solve $\frac{dy}{dx}+\frac{2y}{x}=-x^2y^2\cos x$

Solution: Dividing both the sides of differential equation by y^2

$$\frac{1}{y^2}\frac{dy}{dx}+\frac{2}{x}\frac{1}{y}=-x^2\cos x-----------(1)$$

Put $\frac{1}{y}=V,\quad \therefore -\frac{1}{y^2}\frac{dy}{dx}=\frac{dV}{dx}\ \ or\ \ \frac{1}{y^2}\frac{dy}{dx}=-\frac{dV}{dx}$

Therefore, from (1) $-\frac{dV}{dx}+\frac{2}{x}V=-x^2\cos x$

Multiplying throughout by -1

$$\frac{dV}{dx}-\frac{2}{x}V=x^2\cos x$$

This is a first order linear differential equation of the form $\frac{dV}{dx}+P(x)V=Q(x)$

where, $P(x)=-\frac{2}{x}$ and $Q(x)=x^2\cos x$

$$\int P(x)dx=\int -\frac{2}{x}dx=-2\log x=\log x^{-2}$$

$$\therefore I.F=e^{\int P(x)dx}=e^{\log x^{-2}}=x^{-2}=\frac{1}{x^2}$$

Its solution is $V\cdot(I.F)=\int (I.F)\cdot Q(x)dx+C$

$$\therefore V\cdot\frac{1}{x^2}=\int \frac{1}{x^2}\cdot(x^2\cos x)dx+C$$

$$\frac{1}{y}\cdot\frac{1}{x^2}=\int \cos x dx+C \qquad \left\{\because V=\frac{1}{y}\right\}$$

$$\therefore \boxed{\frac{1}{y}\cdot\frac{1}{x^2}=\sin x+C}$$

Example 5.6.3: Solve $\frac{dy}{dx}-xy=y^2e^{-x^2/2}\log x$

Solution: Dividing both the sides of differential equation by y^2

$$\frac{1}{y^2}\frac{dy}{dx}-x\frac{1}{y}=e^{-x^2/2}\log x-----------(1)$$

Put $\frac{1}{y}=V,\quad \therefore -\frac{1}{y^2}\frac{dy}{dx}=\frac{dV}{dx}\ or\ \frac{1}{y^2}\frac{dy}{dx}=-\frac{dV}{dx}$

Therefore, from (1) $-\frac{dV}{dx}-xV=e^{-x^2/2}\log x$

Multiplying throughout by -1

$$\frac{dV}{dx}+xV=-e^{-x^2/2}\log x$$

This is a first order linear differential equation of the form $\frac{dV}{dx}+P(x)V=Q(x)$

where, $P(x)=x$ and $Q(x)=-e^{-x^2/2}\log x$

$$\int P(x)dx=\int xdx=\frac{x^2}{2}$$

$$\therefore I.F=e^{\int P(x)dx}=e^{x^2/2}$$

Its solution is $V\cdot(I.F)=\int(I.F)\cdot Q(x)dx+C$

$$\therefore V\cdot e^{x^2/2}=\int e^{x^2/2}\cdot\left(-e^{-x^2/2}\log x\right)dx+C$$

$$\frac{1}{y}\cdot e^{x^2/2}=-\int\log xdx+C\qquad\left\{\because V=\frac{1}{y}\right\}$$

$$\therefore \boxed{\frac{1}{y}\cdot e^{x^2/2}=-[x\log x-x]+C}$$

Example 5.6.4: Solve $2\frac{dy}{dx}+y\tan x=\frac{(4x+5)^2}{\cos x}y^3$

Solution: Dividing both the sides of differential equation by y^3

$$\frac{2}{y^3}\frac{dy}{dx}+\tan x\frac{1}{y^2}=\frac{(4x+5)^2}{\cos x}------------(1)$$

Put $\frac{1}{y^2}=V,\quad \therefore -\frac{2}{y^3}\frac{dy}{dx}=\frac{dV}{dx}\quad or\quad \frac{2}{y^3}\frac{dy}{dx}=-\frac{dV}{dx}$

Therefore, from (1) $-\frac{dV}{dx}+(\tan x)V=\frac{(4x+5)^2}{\cos x}$

Multiplying throughout by -1

$$\frac{dV}{dx}-(\tan x)V=-\frac{(4x+5)^2}{\cos x}$$

This is a first order linear differential equation of the form $\frac{dV}{dx}+P(x)V=Q(x)$

where, $P(x)=-\tan x$ and $Q(x)=-\frac{(4x+5)^2}{\cos x}$

$$\int P(x)dx=\int -\tan x dx=-\log\sec x=\log(\sec x)^{-1}=\log(\cos x)$$

$$\therefore I.F=e^{\int P(x)dx}=e^{\log(\cos x)}=\cos x$$

Its solution is $V\cdot(I.F)=\int(I.F)\cdot Q(x)dx+C$

$$\therefore V\cdot\cos x=\int\cos x\cdot\left(-\frac{(4x+5)^2}{\cos x}\right)dx+C$$

$$\therefore \frac{1}{y^2}\cdot\cos x=-\int(4x+5)^2dx+C \qquad \left\{\because V=\frac{1}{y^2}\right\}$$

$$\therefore \frac{1}{y^2}\cdot\cos x=-\frac{(4x+5)^3}{4(3)}+C \qquad \left\{\because \int(ax+b)^n dx=\frac{(ax+b)^{n+1}}{a(n+1)}\right\}$$

$$\therefore \boxed{\frac{1}{y^2}\cdot\cos x=-\frac{(4x+5)^3}{12}+C}$$

Example 5.6.5: Solve $\frac{dy}{dx} = x^3y^3 - xy$

Solution: Writing given differential equation as $\frac{dy}{dx} + xy = x^3y^3$

Dividing both the sides of differential equation by y^3

$$\frac{1}{y^3}\frac{dy}{dx} + x\frac{1}{y^2} = x^3 ------------(1)$$

Put $\frac{1}{y^2} = V, \quad \therefore -\frac{2}{y^3}\frac{dy}{dx} = \frac{dV}{dx} \quad or \quad \frac{1}{y^3}\frac{dy}{dx} = -\frac{1}{2}\frac{dV}{dx}$

Therefore, from (1) $-\frac{1}{2}\frac{dV}{dx} + xV = x^3$

Multiplying throughout by -2

$$\frac{dV}{dx} - 2xV = -2x^3$$

This is a first order linear differential equation of the form $\frac{dV}{dx} + P(x)V = Q(x)$

where, $P(x) = -2x$ and $Q(x) = -2x^3$

$$\int P(x)\,dx = \int -2x\,dx = -x^2 \qquad \therefore I.F = e^{\int P(x)dx} = e^{-x^2}$$

Its solution is $V\cdot(I.F) = \int (I.F)\cdot Q(x)\,dx + C$

$$\therefore V\cdot e^{-x^2} = \int e^{-x^2}\cdot\left(-2x^3\right)dx + C$$

$$\frac{1}{y^2}\cdot e^{-x^2} = \int x^2e^{-x^2}(-2x\,dx) + C \qquad \left\{\because V = \frac{1}{y^2}\right\}$$

On RHS, Put $-x^2 = t \quad \therefore -2x\,dx = dt$

$$\frac{1}{y^2}\cdot e^{-x^2} = \int (-t)e^t\,dt + C$$

$$\frac{1}{y^2}\cdot e^{-x^2} = -\left[t(e^t) - (1)(e^t)\right] + C \qquad \left\{\text{Using } \int uv\,dx = uv_1 - u'v_2 + u''v_3\cdots\right\}$$

$$\frac{1}{y^2}\cdot e^{-x^2} = -\left[-x^2 - 1\right]e^{-x^2} + C \qquad \left\{\because t = -x^2\right\}$$

$$\boxed{\frac{1}{y^2}\cdot e^{-x^2} = \left[x^2 + 1\right]e^{-x^2} + C}$$

Example 5.6.6: Solve $xy(1+xy^2)\frac{dy}{dx}=1$

Solution: Writing given differential equation as $\frac{dx}{dy}-xy=x^2y^3$

Dividing both the sides of differential equation by x^2

$$\frac{1}{x^2}\frac{dx}{dy}-y\frac{1}{x}=y^3 ------------(1)$$

Put $\frac{1}{x}=V, \quad \therefore -\frac{1}{x^2}\frac{dx}{dy}=\frac{dV}{dy} \quad or \quad \frac{1}{x^2}\frac{dx}{dy}=-\frac{dV}{dy}$

Therefore, from (1) $-\frac{dV}{dy}-yV=y^3$

Multiplying throughout by -1, $\quad \frac{dV}{dy}+yV=-y^3$

This is a first order linear differential equation of the form $\frac{dV}{dy}+P(y)V=Q(y)$

where, $P(y)=y$ and $Q(x)=-y^3$

$$\int P(y)dy=\int ydy=\frac{y^2}{2} \qquad \therefore I.F=e^{\int P(y)dy}=e^{y^2/2}$$

Its solution is $V\cdot(I.F)=\int (I.F)\cdot Q(y)dy+C$

$$\therefore V\cdot e^{y^2/2}=\int e^{y^2/2}\cdot(-y^3)dy+C$$

$$\therefore \frac{1}{x}\cdot e^{y^2/2}=-\int y^2e^{y^2/2}\cdot ydy+C \qquad \left\{\because V=\frac{1}{x}\right\}$$

On RHS, Put $\frac{y^2}{2}=t \quad \therefore ydy=dt$

$$\frac{1}{x}\cdot e^{y^2/2}=-\int (2t)e^t dt+C$$

$$\frac{1}{x}\cdot e^{y^2/2}=-2\left[t(e^t)-(1)(e^t)\right]+C \quad \left\{\text{Using } \int uvdx=uv_1-u'v_2+u''v_3\cdots\right\}$$

$$\frac{1}{x}\cdot e^{y^2/2}=-2\left[\frac{y^2}{2}-1\right]e^{y^2/2}+C \quad \left\{\because t=\frac{y^2}{2}\right\}$$

$$\boxed{\frac{1}{x}\cdot e^{y^2/2}=\left[2-y^2\right]e^{y^2/2}+C}$$

Example 5.6.7: Solve $y\frac{dy}{dx}+\frac{4x}{3}-\frac{y^2}{3x}=0$

Solution: Writing given differential equation as $y\frac{dy}{dx}-\frac{1}{3x}y^2=-\frac{4x}{3}$ ------(1)

Put $y^2=V, \quad \therefore 2y\frac{dy}{dx}=\frac{dV}{dx} \quad or \quad y\frac{dy}{dx}=\frac{1}{2}\frac{dV}{dx}$

Therefore, from (1) $\quad \frac{1}{2}\frac{dV}{dx}-\frac{1}{3x}V=-\frac{4x}{3}$

Multiplying throughout by 2

$$\frac{dV}{dx}-\frac{2}{3x}V=-\frac{8x}{3}$$

This is a first order linear differential equation of the form $\frac{dV}{dx}+P(x)V=Q(x)$

where, $P(x)=-\frac{2}{3x}$ and $Q(x)=-\frac{8x}{3}$

$$\int P(x)\,dx=\int -\frac{2}{3x}\,dx=-\frac{2}{3}\log x=\log x^{-2/3}$$

$$\therefore I.F=e^{\int P(x)dx}=e^{\log x^{-2/3}}=x^{-2/3}$$

Its solution is $V\cdot(I.F)=\int (I.F)\cdot Q(x)\,dx+C$

$$\therefore V\cdot x^{-2/3}=\int x^{-2/3}\cdot\left(-\frac{8x}{3}\right)dx+C$$

$$\therefore y^2\cdot x^{-2/3}=\int -\frac{8}{3}x^{1/3}dx+C \qquad \{\because V=y^2\}$$

$$\therefore y^2\cdot x^{-2/3}=-\frac{8}{3}\cdot\frac{x^{4/3}}{4/3}+C$$

$$\therefore \boxed{y^2\cdot x^{-2/3}=-2x^{4/3}+C}$$

Example 5.6.8: Solve $\sec^2 y\frac{dy}{dx}+2\tan x\cdot\tan y=\sin x$

Solution: Writing given differential equation as

$$\sec^2 y\frac{dy}{dx}+(2\tan x)\cdot\tan y=\sin x-----------(1)$$

Put $\tan y=V$, $\therefore \sec^2 y\frac{dy}{dx}=\frac{dV}{dx}$

Therefore, from (1) $\frac{dV}{dx}+(2\tan x)\cdot V=\sin x$

This is a first order linear differential equation of the form $\frac{dV}{dx}+P(x)V=Q(x)$

where, $P(x)=2\tan x$ and $Q(x)=\sin x$

$$\int P(x)dx=\int 2\tan x\ dx=2\log(\sec x)=\log\sec^2 x$$

$$\therefore I.F=e^{\int P(x)dx}=e^{\log\sec^2 x}=\sec^2 x$$

Its solution is $V\cdot(I.F)=\int (I.F)\cdot Q(x)dx+C$

$$\therefore V\cdot\sec^2 x=\int\sec^2 x\cdot(\sin x)dx+C$$

$$\therefore \tan y\cdot\sec^2 x=\int\frac{1}{\cos x}\cdot\left(\frac{\sin x}{\cos x}\right)dx+C\quad\{\because V=\tan y\}$$

$$\therefore \tan y\cdot\sec^2 x=\int\sec x\cdot\tan x dx+C$$

$$\therefore \boxed{\tan y\cdot\sec^2 x=\sec x+C}$$

Example 5.6.9: $\frac{dy}{dx} = e^{x-y}(e^x - e^y)$

Solution: Writing given differential equation as $\frac{dy}{dx} = e^{2x-y} - e^x$

$$\frac{dy}{dx} + e^x = \frac{e^{2x}}{e^y}$$

$$e^y \frac{dy}{dx} + e^x \cdot e^y = e^{2x} ------------(1)$$

Put $e^y = V$, $\therefore e^y \frac{dy}{dx} = \frac{dV}{dx}$

Therefore, from (1) $\frac{dV}{dx} + e^x \cdot V = e^{2x}$

This is a first order linear differential equation of the form $\frac{dV}{dx} + P(x)V = Q(x)$

where, $P(x) = e^x$ and $Q(x) = e^{2x}$

$$\int P(x)\,dx = \int e^x \; dx = e^x$$

$$\therefore I.F = e^{\int P(x)dx} = e^{e^x}$$

Its solution is $V \cdot (I.F) = \int (I.F) \cdot Q(x)\,dx + C$

$$\therefore V \cdot e^{e^x} = \int e^{e^x} \cdot \left(e^{2x}\right) dx + C$$

$$\therefore e^y \cdot e^{e^x} = \int e^x e^{e^x} \cdot \left(e^x dx\right) dx + C \qquad \{\because V = e^y\}$$

On RHS, Put $e^x = t$ $\therefore e^x dx = dt$

$$e^y \cdot e^{e^x} = \int t e^t dt + C$$

$$e^y \cdot e^{e^x} = \left[t(e^t) - (1)(e^t)\right] + C \quad \left\{\text{Using } \int uv dx = uv_1 - u'v_2 + u''v_3 \cdots\right\}$$

$$\therefore \boxed{e^y \cdot e^{e^x} = \left[e^x - 1\right] e^{e^x} + C} \quad \{\because t = e^x\}$$

Example 5.6.10: Solve $\frac{dy}{dx}+(2x\tan^{-1}y-x^3)(1+y^2)=0$

Solution: Writing given differential equation as $\frac{1}{1+y^2}\frac{dy}{dx}+2x\tan^{-1}y-x^3=0$

$$\therefore \frac{1}{1+y^2}\frac{dy}{dx}+2x\tan^{-1}y=x^3 \text{ -----------(1)}$$

Put $\tan^{-1}y=V$, $\therefore \frac{1}{1+y^2}\frac{dy}{dx}=\frac{dV}{dx}$

Therefore, from (1) $\frac{dV}{dx}+(2x)\cdot V=x^3$

This is a first order linear differential equation of the form $\frac{dV}{dx}+P(x)V=Q(x)$

where, $P(x)=2x$ and $Q(x)=x^3$

$$\int P(x)dx=\int 2x\,dx=x^2$$

$$\therefore I.F=e^{\int P(x)dx}=e^{x^2}$$

Its solution is $V\cdot(I.F)=\int(I.F)\cdot Q(x)dx+C$

$$\therefore V\cdot e^{x^2}=\int e^{x^2}\cdot(x^3)dx+C$$

$$\therefore (\tan^{-1}y)\cdot e^{x^2}=\int e^{x^2}\cdot(x^3)dx+C \quad \{\because V=\tan^{-1}y\}$$

$$\therefore (\tan^{-1}y)\cdot e^{x^2}=\int x^2e^{x^2}\cdot(xdx)+C$$

On RHS, putting $x^2=t$, $\therefore 2xdx=dt$

$$\therefore (\tan^{-1}y)\cdot e^{x^2}=\int te^t\cdot\left(\frac{dt}{2}\right)+C$$

$$\therefore (\tan^{-1}y)\cdot e^{x^2}=\frac{1}{2}\int te^t dt+C$$

$$\therefore (\tan^{-1}y)\cdot e^{x^2}=\frac{1}{2}\left[(t)e^t-(1)e^t\right]+C \quad \left\{\text{Using } \int uvdx=uv_1-u'v_2+u''v_3\cdots\right\}$$

$$\therefore \boxed{(\tan^{-1}y)\cdot e^{x^2}=\frac{1}{2}(x^2-1)e^{x^2}+C}$$

Example 5.6.11: Solve $\frac{dr}{d\theta}=\frac{r\sin\theta-r^2}{\cos\theta}$

Solution: Writing given differential equation as $\frac{dr}{d\theta}-\frac{\sin\theta}{\cos\theta}r=-\frac{r^2}{\cos\theta}$

$$\frac{dr}{d\theta}-(\tan\theta)r=-r^2\sec\theta$$

Dividing by $-r^2$

$$-\frac{1}{r^2}\frac{dr}{d\theta}+(\tan\theta)\frac{1}{r}=\sec\theta------------(1)$$

Put $\frac{1}{r}=V,\quad \therefore -\frac{1}{r^2}\frac{dr}{d\theta}=\frac{dV}{d\theta}$

Therefore, from (1) $\frac{dV}{d\theta}+(\tan\theta)\cdot V=\sec\theta$

This is a first order linear differential equation of the form $\frac{dV}{d\theta}+P(\theta)V=Q(\theta)$

where, $P(\theta)=\tan\theta$ and $Q(\theta)=\sec\theta$

$$\int P(\theta)\,d\theta=\int\tan\theta\,d\theta=\log\sec\theta$$

$$\therefore I.F=e^{\int P(\theta)d\theta}=e^{\log\sec\theta}=\sec\theta$$

Its solution is $V\cdot(I.F)=\int(I.F)\cdot Q(\theta)\,d\theta+C$

$$\therefore V\cdot\sec\theta=\int\sec\theta\cdot(\sec\theta)\,d\theta+C$$

$$\therefore \frac{1}{r}\cdot\sec\theta=\int\sec^2\theta\cdot d\theta+C\quad\left\{\because V=\frac{1}{r}\right\}$$

$$\therefore \frac{1}{r}\cdot\sec\theta=\tan\theta+C$$

$$\therefore \boxed{\frac{1}{r}\cdot\sec\theta=\tan\theta+C}$$

Example 5.6.12: Solve $\frac{dy}{dx}+\frac{y}{x}\log y=\frac{y}{x^2}(\log y)^2$

Solution: Dividing both the sides of differential equation by $y(\log y)^2$

$$\frac{dy}{dx}+\frac{y}{x}\log y=\frac{y}{x^2}(\log y)^2$$

$$\frac{1}{y(\log y)^2}\frac{dy}{dx}+\frac{1}{x}\cdot\frac{1}{\log y}=\frac{1}{x^2}------------(1)$$

Put $\frac{1}{\log y}=V$, $\therefore -\frac{1}{(\log y)^2}\frac{1}{y}\frac{dy}{dx}=\frac{dV}{dx}$ or $\frac{1}{y(\log y)^2}\frac{dy}{dx}=-\frac{dV}{dx}$

Therefore, from (1) $-\frac{dV}{dx}+\frac{1}{x}\cdot V=\frac{1}{x^2}$

Multiplying throughout by -1

$$\frac{dV}{dx}-\frac{1}{x}\cdot V=-\frac{1}{x^2}$$

This is a first order linear differential equation of the form $\frac{dV}{dx}+P(x)V=Q(x)$

where, $P(x)=-\frac{1}{x}$ and $Q(x)=\frac{1}{x^2}$

$$\int P(x)\,dx=\int -\frac{1}{x}\,dx=-\log x=\log x^{-1}$$

$$\therefore I.F=e^{\int P(x)dx}=e^{\log x^{-1}}=x^{-1}=\frac{1}{x}$$

Its solution is $V\cdot(I.F)=\int (I.F)\cdot Q(x)\,dx+C$

$$\therefore V\cdot\frac{1}{x}=\int\frac{1}{x}\cdot\left(\frac{1}{x^2}\right)dx+C$$

$$\therefore \frac{1}{\log y}\cdot\frac{1}{x}=\int\frac{1}{x^3}dx+C \quad \left\{\because V=\frac{1}{\log y}\right\}$$

$$\therefore \frac{1}{\log y}\cdot\frac{1}{x}=\frac{x^{-3+1}}{-3+1}+C$$

$$\therefore \boxed{\frac{1}{x\log y}=-\frac{1}{2x^2}+C}$$

Example 5.6.13: Solve $\frac{dy}{dx}=1-x(y-x)-x^3(y-x)^2$

Solution: Writing given differential equation as $\left(\frac{dy}{dx}-1\right)+x(y-x)=-x^3(y-x)^2$

Put $y-x=Z, \quad \therefore \frac{dy}{dx}-1=\frac{dZ}{dx}$

$\therefore \frac{dZ}{dx}+xZ=-x^3Z^2$

Dividing both the sides of differential equation by Z^2

$\frac{1}{Z^2}\frac{dZ}{dx}+x\frac{1}{Z}=-x^3 ------------(1)$

Put $\frac{1}{Z}=V, \quad \therefore -\frac{1}{Z^2}\frac{dZ}{dx}=\frac{dV}{dx} \; or \; \frac{1}{Z^2}\frac{dZ}{dx}=-\frac{dV}{dx}$

Therefore, from (1) $-\frac{dV}{dx}+xV=-x^3$

Multiplying throughout by -1, $\quad \frac{dV}{dx}-xV=x^3$

This is a first order linear differential equation of the form $\frac{dV}{dx}+P(x)V=Q(x)$

where, $P(x)=-x$ and $Q(x)=x^3$

$\int P(x)dx=\int -xdx=-\frac{x^2}{2} \qquad \therefore I.F=e^{\int P(x)dx}=e^{-x^2/2}$

Its solution is $V\cdot(I.F)=\int (I.F)\cdot Q(x)dx+C$

$\therefore V\cdot e^{-x^2/2}=\int e^{-x^2/2}\cdot\left(x^3\right)dx+C$

$\therefore \frac{1}{y-x}\cdot e^{-x^2/2}=\int x^2e^{-x^2/2}(xdx)+C \qquad \left\{\because V=\frac{1}{Z}=\frac{1}{y-x}\right\}$

On RHS, Put $\frac{x^2}{2}=t \quad \therefore xdx=dt$

$\therefore \frac{1}{y-x}\cdot e^{-x^2/2}=\int (2t)e^{-t}dt+C=2\left[t(-e^{-t})-(1)(e^{-t})\right]+C$ { Integrating by parts}

$$\therefore \boxed{\frac{1}{y-x}\cdot e^{-x^2/2}=-2\left[\frac{x^2}{2}+1\right]e^{-x^2/2}+C}$$

Example 5.6.14: Solve $(1+\sin y)\dfrac{dx}{dy}=[2y\cos y-x(\sec y+\tan y)]$

Solution: Writing given differential equation as $(1+\sin y)\dfrac{dx}{dy}+(\sec y+\tan y)x=2y\cos y$

Dividing both the sides of differential equation by $1+\sin y$

$$\frac{dx}{dy}+\left(\frac{\sec y+\tan y}{1+\sin y}\right)x=\left(\frac{2y\cos y}{1+\sin y}\right)$$

This is a first order linear differential equation of the form $\dfrac{dx}{dy}+P(y)x=Q(y)$

where, $P(y)=\dfrac{\sec y+\tan y}{1+\sin y}=\left(\dfrac{1}{\cos y}+\dfrac{\sin y}{\cos y}\right)\cdot\dfrac{1}{1+\sin y}=\dfrac{\cancel{1+\sin y}}{\cos y}\cdot\dfrac{1}{\cancel{1+\sin y}}=\dfrac{1}{\cos y}=\sec y$

and $Q(y)=\dfrac{2y\cos y}{1+\sin y}$

$$\int P(y)\,dy=\int \sec y\,dy=\log(\sec y+\tan y)$$

$$\therefore I.F=e^{\int P(y)dy}=e^{\log(\sec y+\tan y)}=\sec y+\tan y=\frac{1}{\cos y}+\frac{\sin y}{\cos y}=\frac{1+\sin y}{\cos y}$$

Its solution is $x\cdot(I.F)=\int(I.F)\cdot Q(y)\,dy+C$

$$\therefore x\cdot\left(\frac{1+\sin y}{\cos y}\right)=\int\left(\frac{1+\sin y}{\cos y}\right)\cdot\left(\frac{2y\cos y}{1+\sin y}\right)dy+C$$

$$\therefore x\cdot\left(\frac{1+\sin y}{\cos y}\right)=\int 2y\,dy+C$$

$$\therefore \boxed{x\cdot\left(\frac{1+\sin y}{\cos y}\right)=y^2+C}$$

Example 5.6.15: Solve $\frac{dy}{dx}+x^3\sin^2 y+x\sin 2y=x^3$

Solution: Writing given differential equation as $\frac{dy}{dx}+x\sin 2y=x^3\left(1-\sin^2 y\right)$

$$\therefore \frac{dy}{dx}+x\sin 2y=x^3\cos^2 y$$

Dividing both the sides of differential equation by $\cos^2 y$

$$\therefore \frac{1}{\cos^2 y}\frac{dy}{dx}+x\left(\frac{2\sin y\cos y}{\cos^2 y}\right)=x^3$$

$$\therefore \sec^2 y\frac{dy}{dx}+x\left(2\tan y\right)=x^3 --------(1)$$

Put $\tan y=V$, $\therefore \sec^2 y\frac{dy}{dx}=\frac{dV}{dx}$

Therefore, from (1) $\frac{dV}{dx}+2xV=x^3$

This is a first order linear differential equation of the form $\frac{dV}{dx}+P\left(x\right)V=Q\left(x\right)$

where, $P\left(x\right)=2x$ and $Q\left(x\right)=x^3$

$$\int P\left(x\right)dx=\int 2x\ dx=x^2 \qquad \therefore I.F=e^{\int P(x)dx}=e^{x^2}$$

Its solution is $V\cdot\left(I.F\right)=\int\left(I.F\right)\cdot Q\left(x\right)dx+C$

$$\therefore V\cdot e^{x^2}=\int e^{x^2}\cdot\left(x^3\right)dx+C$$

$$\therefore \left(\tan y\right)\cdot e^{x^2}=\int e^{x^2}\cdot\left(x^3\right)dx+C \quad \left\{\because V=\tan y\right\}$$

$$\therefore \left(\tan y\right)\cdot e^{x^2}=\int x^2e^{x^2}\cdot\left(xdx\right)+C$$

On RHS, putting $x^2=t$, $\therefore 2xdx=dt$

$$\therefore \left(\tan y\right)\cdot e^{x^2}=\int te^t\cdot\left(\frac{dt}{2}\right)+C=\frac{1}{2}\int te^t dt+C$$

$$\therefore \left(\tan y\right)\cdot e^{x^2}=\frac{1}{2}\left[(t)e^t-(1)e^t\right]+C$$

$$\therefore \boxed{\left(\tan y\right)\cdot e^{x^2}=\frac{1}{2}(x^2-1)e^{x^2}+C}$$